

HOMOTOPY INVARIANCE THROUGH SMALL STABILIZATIONS

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ABSTRACT. We associate an algebra $\Gamma^\infty(\mathfrak{A})$ to each bornological algebra \mathfrak{A} . The algebra $\Gamma^\infty(\mathfrak{A})$ contains a two-sided ideal $I_{S(\mathfrak{A})}$ for each symmetric ideal $S \triangleleft \ell^\infty$ of bounded sequences of complex numbers. In the case of $\Gamma^\infty = \Gamma^\infty(\mathbb{C})$, these are all the two-sided ideals, and $I_S \mapsto J_S = \mathcal{B}I_S\mathcal{B}$ gives a bijection between the two-sided ideals of Γ^∞ and those of $\mathcal{B} = \mathcal{B}(\ell^2)$. We prove that Weibel's K -theory groups $KH_*(I_{S(\mathfrak{A})})$ are homotopy invariant for certain ideals S including c_0 and ℓ^p . Moreover, if either $S = c_0$ and \mathfrak{A} is a local C^* -algebra or $S = \ell^p, \ell^{p\pm}$ and \mathfrak{A} is a local Banach algebra, then $KH_*(I_{S(\mathfrak{A})})$ contains $K_*^{\text{top}}(\mathfrak{A})$ as a direct summand. Furthermore, we prove that for $S \in \{c_0, \ell^p, \ell^{p\pm}\}$ there is a long exact sequence

$$\begin{array}{ccc} KH_{n+1}(I_{S(\mathfrak{A})}) & \longrightarrow & HC_{n-1}(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) \\ & & \downarrow \\ KH_n(I_{S(\mathfrak{A})}) & \longleftarrow & K_n(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) \end{array}$$

For $\mathfrak{A} = \mathbb{C}$ we compare this sequence to an analogous sequence for J_S and show that the map $KH_n(I_S) \rightarrow KH_n(J_S)$ is surjective for all $n \geq 0$ and that $HC_n(\Gamma^\infty : I_S) \rightarrow HC_n(\mathcal{B} : J_S)$ is an isomorphism for certain values of n .

1. INTRODUCTION

Let $\ell^2 = \ell^2(\mathbb{N})$ be the Hilbert space of square-summable sequences of complex numbers and $\mathcal{B} = \mathcal{B}(\ell^2)$ the algebra of bounded operators. Let Emb be the inverse monoid of all partially defined injections

$$\mathbb{N} \supset \text{dom} f \xrightarrow{f} \mathbb{N}.$$

Each element $f \in \text{Emb}$ defines a partial isometry $U_f \in \mathcal{B}$; for the canonical Hilbert basis we have $U_f(e_n) = e_{f(n)}$ if $n \in \text{dom} f$ and 0 otherwise. Similarly, each bounded sequence of complex numbers $\alpha \in \ell^\infty$ defines an element $\text{diag}(\alpha) \in \mathcal{B}$ by $\text{diag}(\alpha)(e_n) = \alpha_n e_n$. The subspace generated by all the U_f and $\text{diag}(\alpha)$ for $f \in \text{Emb}$ and $\alpha \in \ell^\infty$ is the subalgebra

$$\mathcal{B} \supset \Gamma^\infty := \text{span}\{\text{diag}(\alpha)U_f : \alpha \in \ell^\infty, f \in \text{Emb}\}.$$

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In this article we show that the algebra Γ^∞ has several remarkable properties. One of them is that the lattice of two-sided ideals of Γ^∞ is isomorphic to the lattice of two-sided ideals of \mathcal{B} . A theorem of Calkin, as restated by Garling, establishes a one-to-one correspondence between two-sided ideals of \mathcal{B} and the ideals of ℓ^∞ that are symmetric, that is, invariant under the action of Emb . Calkin's correspondence maps a symmetric ideal $S \triangleleft \ell^\infty$ to the ideal J_S of those operators whose sequence of singular values belongs to S . Consider the subspace

$$\Gamma^\infty \supset I_S := \text{span}\{\text{diag}(\alpha)U_f : \alpha \in S, f \in \text{Emb}\}.$$

Note that $I_{\ell^\infty} = \Gamma^\infty$; for all symmetric ideals S , $I_S \triangleleft \Gamma^\infty$ is a two-sided ideal. We prove (see Theorem 4.5)

Theorem 1.1. *The map $J \mapsto J \cap \Gamma^\infty$ is a bijection between the sets of two-sided ideals of $\mathcal{B}(\ell^2(\mathbb{N}))$ and Γ^∞ . If $S \triangleleft \ell^\infty$ is a symmetric ideal, then $J_S \cap \Gamma^\infty = I_S$.*

More generally, we define for any bornological algebra \mathfrak{A} (in particular for a Banach algebra \mathfrak{A}) an algebra $\Gamma^\infty(\mathfrak{A})$. The algebra $\Gamma^\infty(\mathfrak{A})$ contains an ideal $I_{S(\mathfrak{A})}$ for any symmetric ideal $S \triangleleft \ell^\infty$, and $S \mapsto I_{S(\mathfrak{A})}$ is a lattice homomorphism. Thus the smallest nonzero $I_{S(\mathfrak{A})}$ occurs when S is the symmetric ideal $c_f \triangleleft \ell^\infty$ of finitely supported sequences; we get

$$I_{c_f}(\mathfrak{A}) = M_\infty \mathfrak{A} = \bigcup_n M_n \mathfrak{A}.$$

Hence the inclusion $\mathfrak{A} \rightarrow M_\infty \mathfrak{A}$ into the upper left corner gives a stability homomorphism

$$\iota_S : \mathfrak{A} \rightarrow I_{c_f}(\mathfrak{A}) \subset I_{S(\mathfrak{A})}.$$

If \mathfrak{A} is unital then ι_{c_f} induces an isomorphism in algebraic K -theory, by matrix stability. At the other extreme $I_{\ell^\infty(\mathfrak{A})} = \Gamma^\infty(\mathfrak{A})$ is a ring with infinite sums in the sense of [28] (see Proposition 6.1.6); this permits the Eilenberg swindle and we have

$$K_*(\Gamma^\infty(\mathfrak{A})) = 0.$$

For $c_f \subsetneq S \subsetneq \ell^\infty$, the K -theory of $I_S(\mathfrak{A})$ is more interesting. We study it for

$$S \in \{c_0, \ell^{p-}, \ell^q, \ell^{q+} \mid (p \leq \infty, q < \infty)\}. \quad (1.2)$$

Here c_0 is the ideal of sequences vanishing at infinity, ℓ^q consists of the q -summable sequences, and

$$\ell^{p-} = \bigcup_{r < p} \ell^r, \quad \ell^{q+} = \bigcap_{s > q} \ell^s.$$

Let BAlg be the category of bornological algebras. We consider several variants of K -theory. We write K for algebraic K -theory, KH for Weibel's homotopy algebraic K -theory and K^{top} for topological K -theory. The following result follows from Theorems 8.1.9 and 10.1.9(i).

Theorem 1.3.

- i) The functor $\mathbf{BAlg} \rightarrow \mathfrak{Ab}$, $\mathfrak{A} \mapsto KH_*(I_{c_0(\mathfrak{A})})$ is invariant under continuous homotopy.
- ii) If \mathfrak{A} is a local C^* -algebra and $n \geq 0$, then there is a natural split monomorphism

$$K_n^{\text{top}}(\mathfrak{A}) \twoheadrightarrow KH_n(I_{c_0(\mathfrak{A})}) .$$

- iii) If $n \leq 0$, then the comparison map

$$K_n(I_{c_0(\mathfrak{A})}) \rightarrow KH_n(I_{c_0(\mathfrak{A})}) \quad (1.4)$$

is an isomorphism for every $\mathfrak{A} \in \mathbf{BAlg}$. If furthermore \mathfrak{A} is a C^* -algebra, then (1.4) is an isomorphism for all $n \in \mathbb{Z}$.

The results above should be compared with Karoubi's conjecture (Suslin-Wodzicki's theorem [26, Theorem 10.9]) that for a C^* -algebra \mathfrak{A} , the comparison map

$$K_*(\mathfrak{A} \otimes \tilde{\mathcal{K}}) \rightarrow K_*^{\text{top}}(\mathfrak{A} \otimes \tilde{\mathcal{K}}) \cong K_*^{\text{top}}(\mathfrak{A})$$

is an isomorphism. Hence we may think of $\mathfrak{A} \rightarrow I_{c_0(\mathfrak{A})}$ as a smaller version of the stabilization $\mathfrak{A} \mapsto \mathfrak{A} \otimes \tilde{\mathcal{K}}$ whose K -theory is homotopy invariant and contains $K_*^{\text{top}}(\mathfrak{A})$ as a direct summand. Next let $p \geq 1$ and consider the Schatten ideal $\mathcal{L}^p \triangleleft \mathcal{B}$. Remark that \mathcal{L}^p is the ideal corresponding to ℓ^p under Calkin's correspondence. We have

$$\mathcal{L}^p = J_{\ell^p}.$$

Recall from [10, Theorem 6.2.1] that if \mathfrak{A} is a locally convex algebra and $\mathfrak{A} \hat{\otimes} \mathcal{L}^p$ is the projective tensor product then

$$KH_*(\mathfrak{A} \hat{\otimes} \mathcal{L}^1) \xrightarrow{\cong} KH_*(\mathfrak{A} \hat{\otimes} \mathcal{L}^p) \xrightarrow{\cong} K_*^{\text{top}}(\mathfrak{A} \hat{\otimes} \mathcal{L}^p).$$

In the present article (Theorems 8.1.1 and 10.1.9(ii)) we prove the following analogue of the latter result.

Theorem 1.5. *Let S be one of ℓ^p , ℓ^{p+} ($0 < p < \infty$) or ℓ^{p-} ($0 < p \leq \infty$).*

- i) The functor $\mathbf{BAlg} \rightarrow \mathfrak{Ab}$, $\mathfrak{A} \mapsto KH_*(I_{\ell^1(\mathfrak{A})})$ is invariant under Hölder-continuous homotopies and we have $KH_*(I_S(\mathfrak{A})) = KH_*(I_{\ell^1(\mathfrak{A})})$ for all S as above.
- ii) If \mathfrak{A} is a local Banach algebra and $n \geq 0$, then there is a natural split monomorphism

$$K_n^{\text{top}}(\mathfrak{A}) \twoheadrightarrow KH_n(I_{\ell^1(\mathfrak{A})}) .$$

- iii) If $n \leq 0$, then the comparison map

$$K_n(I_S(\mathfrak{A})) \rightarrow KH_n(I_S(\mathfrak{A})) \quad (1.6)$$

is an isomorphism for every $\mathfrak{A} \in \mathbf{BAlg}$. If furthermore \mathfrak{A} is a unital Banach algebra and $S = \ell^{\infty-}$ then (1.6) is an isomorphism for all $n \in \mathbb{Z}$.

Both these theorems rely on a homotopy invariance theorem (Theorem 7.4.1) which we think is of independent interest. The theorem says that if $F : \mathbb{C}\text{-Alg} \rightarrow \mathfrak{A}\mathfrak{b}$ is an M_2 -stable, split exact functor and $S \in \{c_0, \ell^p\}$ then the functor

$$\mathbf{BAlg} \rightarrow \mathfrak{A}\mathfrak{b}, \quad \mathfrak{A} \mapsto F(I_{S(\mathfrak{A})})$$

is homotopy invariant. For $S = c_0$ it is continuous homotopy invariant, while for $S = \ell^p$ it is invariant under Hölder continuous homotopies, with Hölder exponent depending on p . For $F = KH_*$ we have $KH_*(I_{\ell^p(\mathfrak{A})}) = KH_*(I_{\ell^1(\mathfrak{A})})$, and so it is invariant under arbitrary Hölder continuous homotopies. Furthermore, we have the following general result (see Theorem 8.2.1) about the comparison map $K \rightarrow KH$. Its proof uses the homotopy invariance theorem mentioned above applied to infinitesimal K -theory.

Theorem 1.7. *Let \mathfrak{A} be a bornological algebra and let S be c_0 , ℓ^p , ℓ^{p+} ($0 < p < \infty$) or ℓ^{p-} ($0 < p \leq \infty$). Then there are long exact sequences ($n \in \mathbb{Z}$)*

$$KH_{n+1}(I_{S(\mathfrak{A})}) \longrightarrow HC_{n-1}(I_{S(\mathfrak{A})}) \quad (1.8)$$

$$\begin{array}{c} \downarrow \\ KH_n(I_{S(\mathfrak{A})}) \longleftarrow K_n(I_{S(\mathfrak{A})}) \end{array}$$

and

$$KH_{n+1}(I_{S(\mathfrak{A})}) \longrightarrow HC_{n-1}(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) \quad (1.9)$$

$$\begin{array}{c} \downarrow \\ KH_n(I_{S(\mathfrak{A})}) \longleftarrow K_n(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) \end{array}$$

The theorem above is applied, for example, to show (in Theorem 10.1.9) that, as asserted in part (iii) of Theorems 1.3 and 1.5, the comparison map $K_*(I_{S(\mathfrak{A})}) \rightarrow KH_*(I_{S(\mathfrak{A})})$ is an isomorphism if either $S = c_0$ and \mathfrak{A} is a C^* -algebra or $S = \ell^{\infty-}$ and \mathfrak{A} is a unital Banach algebra. Indeed the proof of the latter result consists of showing that $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) = 0$ in both of these cases. Next consider the case $\mathfrak{A} = \mathbb{C}$ of the sequence (1.9). By [10, Proposition 7.2.1] or [32, Theorem 5] for S as in Theorem 1.7 there is an analogous exact sequence

$$\begin{array}{ccccc} HC_{2n-1}(\mathcal{B} : J_S) & \longrightarrow & K_{2n}(\mathcal{B} : J_S) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_{2n-1}(\mathcal{B} : J_S) & \longleftarrow & HC_{2n-2}(\mathcal{B} : J_S) \end{array} \quad (1.10)$$

Here we are using the fact that $KH_n(J_S) = K_n^{\text{top}}(\mathbb{C})$. The inclusions $\Gamma^\infty \subset \mathcal{B}$ and $I_S \subset J_S$ induce a map of exact sequences from (1.9) to (1.10). By Remark 8.1.11 when $n \geq 0$ the map

$$KH_n(I_S) \rightarrow KH_n(J_S) = K_n^{\text{top}}(\mathbb{C})$$

is onto for all S as above. Corollary 9.5.3 establishes that the map

$$HC_0(\Gamma^\infty : I_S) \rightarrow HC_0(\mathcal{B} : J_S) \quad (1.11)$$

is an isomorphism for any symmetric ideal $S \triangleleft \ell^\infty$. Next assume

$$S \in \{\ell^p, \ell^{p-}, \ell^{p+} : 0 < p < \infty\}.$$

For such S , the following integer was computed by Wodzicki in [32]

$$m = \min\{n : HC_n(\mathcal{B} : J_S) \neq 0\}.$$

We prove in Proposition 10.1.7 that

$$m = \min\{n : HC_n(\Gamma^\infty : I_S) \neq 0\}, \quad (1.12)$$

and that the natural map is an isomorphism

$$HC_m(\Gamma^\infty : I_S) \xrightarrow{\cong} HC_m(\mathcal{B} : J_S). \quad (1.13)$$

More calculations concerning the groups $HC_*(\Gamma^\infty : I_S)$ for S as above are carried out in Proposition 10.3.3. These calculations employ Theorem 10.2.5, which gives a formula for the groups $HC_*(\Gamma^\infty : I_S)$ for an arbitrary symmetric ideal $S \triangleleft \ell^\infty$, and Corollary 10.2.6, which provides a spectral sequence to compute them. The formula expresses the groups $HC_*(\Gamma^\infty : I_S)$ in terms of the action of Emb on subcomplexes of the algebraic de Rham complex of ℓ^∞ .

The rest of this paper is organized as follows. In Section 2 we establish some notation about sequence spaces, the inverse monoid Emb and the partial isometries U_f . The algebra $\Gamma^\infty(\mathfrak{A})$ and the ideals $I_{S(\mathfrak{A})}$ are introduced in Section 3. In this section we also recall the definition of Karoubi's cone $\Gamma(R)$ which is R -linearly generated by the U_f ($f \in \text{Emb}$). Proposition 3.14 identifies $I_{S(\mathfrak{A})}$ with a ring formed by certain $\mathbb{N} \times \mathbb{N}$ matrices with coefficients in \mathfrak{A} . The two-sided ideals of Γ^∞ are studied in Section 4; Theorem 1.1 is contained in Theorem 4.5. In Section 5 we show that $I_{S(\mathfrak{A})}$ can be written as a crossed product of $\Gamma = \Gamma(\mathbb{Q})$ and $S(\mathfrak{A})$, by using the conjugation action of Emb in $S(\mathfrak{A})$ via the partial isometries U_f (Proposition 5.11). Section 6 contains an assortment of miscellaneous properties of Γ^∞ and related algebras. For example we show that if \mathfrak{A} is unital, then $\Gamma^\infty(\mathfrak{A})$ is a ring with infinite sums in the sense of Wagoner (Proposition 6.1.6). We also prove some flatness results, including that every two-sided ideal of Γ^∞ is flat both as a left and as a right Γ^∞ -module (Proposition 6.2.5). Section 7 deals with the homotopy invariance theorem mentioned above, proved in Theorem 7.4.1. Applications to K -theory are given in Section 8; see Theorems 8.1.1, 8.1.9 and 8.2.1. In Section 5 we study the homology of crossed products with Γ . We give general formulas for the Hochschild and cyclic homology of such products; see Proposition 9.4.4 and Theorem 9.6.3. The fact that (1.11) is an isomorphism is proved in Corollary 9.5.3. Section 10 is devoted to the computation of the relative cyclic homology groups $HC_*(\Gamma^\infty : I_S)$. The

proof of the identity (1.12) and of the isomorphism (1.13) is contained in Proposition 10.1.7, which proves also that

$$HC_*(\Gamma^\infty : I_{c_0}) = HC_*(\Gamma^\infty : I_{\ell^\infty-}) = 0.$$

Theorem 10.1.9 establishes that the comparison maps (1.4) and (1.6) are isomorphisms when \mathfrak{A} is a C^* -algebra and when \mathfrak{A} is a unital Banach algebra, respectively. Next we turn to the case $\mathfrak{A} = \mathbb{C}$. A formula for $HC_*(\Gamma^\infty : I_S)$ and a spectral sequence to compute it are given in Theorem 10.2.5 and Corollary 10.2.6; they both apply to an arbitrary symmetric ideal $S \triangleleft \ell^\infty$. Finally we focus on the case $S \in \{\ell^p, \ell^{p\pm}\}$ and make several computations of the groups $HC_*(\Gamma^\infty : I_S)$ in Proposition 10.3.3.

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2. PRELIMINARIES

2.1. Sequence ideals. Throughout this paper we work in the setting of bornological spaces and bornological algebras; a quick introduction to the subject is given in [13, Chapter 2]. Recall a (complete, convex) bornological vector space over the field \mathbb{C} of complex numbers is a filtering union $\mathbb{V} = \cup_D \mathbb{V}_D$ of Banach spaces, indexed by the disks of \mathbb{V} such that the inclusions $\mathbb{V}_D \subset \mathbb{V}_{D'}$ are bounded. A subset of \mathbb{V} is *bounded* if it is a bounded subset of some \mathbb{V}_D . A sequence $\mathbb{N} \rightarrow \mathbb{V}$ is *bounded* if its image is a bounded subset of \mathbb{V} . We write $\ell^\infty(\mathbb{N}, \mathbb{V})$ or simply $\ell^\infty(\mathbb{V})$ for the bornological vector space of bounded sequences where $X \subset \ell^\infty(\mathbb{V})$ is bounded if $\bigcup_{x \in X} x(\mathbb{N})$ is. We consider the following closed bornological subspace

$$\ell^\infty(\mathbb{V}) \supset c_0(\mathbb{V}) = \{\alpha : \lim_n \alpha_n = 0\} \quad (2.1.1)$$

We also consider the subspace ($p > 0$)

$$c_0(\mathbb{V}) \supset \ell^p(\mathbb{V}) = \{\alpha : \mathbb{N} \rightarrow \mathbb{V} : (\exists \text{ a disk } D \subset \mathbb{V}) \sum_n \|\alpha_n\|_D^p < \infty\}$$

If $p \geq 1$, we equip $\ell^p(\mathbb{V})$ with the following bornology: we say that a subset $S \subset \ell^p(\mathbb{V})$ is bounded if there exist a disk D and a constant C such that $\sum_n \|\alpha_n\|_D^p < C$ for all $\alpha \in S$. Notice that the inclusion $\ell^p(\mathbb{V}) \rightarrow \ell^\infty(\mathbb{V})$ is bounded for $p \geq 1$. Recall a bornological algebra is a bornological vector space \mathfrak{A} with an associative bounded multiplication. If \mathfrak{A} is a bornological algebra, then pointwise multiplication makes $\ell^\infty(\mathfrak{A})$ into a bornological algebra, $c_0(\mathfrak{A}) \triangleleft \ell^\infty(\mathfrak{A})$ is a closed bornological ideal, and $\ell^p(\mathfrak{A}) \triangleleft \ell^\infty(\mathfrak{A})$ is an algebraic ideal for all $p > 0$.

Notation 2.1.2. When \mathfrak{A} is \mathbb{C} , we shall omit it from our notation. Thus we shall write ℓ^∞ , ℓ^p , c_0 , etc, for $\ell^\infty(\mathbb{C})$, $\ell^p(\mathbb{C})$, $c_0(\mathbb{C})$, etc.

The space $\mathcal{B}(\ell^2(\mathbb{V}))$ of bounded operators $\ell^2(\mathbb{V}) \rightarrow \ell^2(\mathbb{V})$ on a bornological vector space \mathbb{V} is a bornological algebra with the uniform bornology ([13, Def. 2.4]). If \mathfrak{A} is a bornological algebra, then

$$\text{diag} : \ell^\infty(\mathfrak{A}) \rightarrow \mathcal{B}(\ell^2(\mathfrak{A})), \quad \text{diag}(\alpha)(\xi) = (\alpha_n \xi_n)_{n \geq 1}. \quad (2.1.3)$$

is a bounded representation. It is faithful if and only if the left annihilator of \mathfrak{A} is trivial:

$$\text{ann}(\mathfrak{A}) = \{a \in \mathfrak{A} : a \cdot b = 0 \quad (\forall b \in \mathfrak{A})\} = 0,$$

This happens, for instance, when \mathfrak{A} is unital.

2.2. The monoid Emb. We begin by recalling some definitions from [15]. We denote by Emb the set of injective functions

$$\text{Emb} = \{f : A \rightarrow \mathbb{N} : A \subset \mathbb{N}\}.$$

Note that Emb is a monoid for the composition law:

$$fg : \text{dom}(g) \cap g^{-1}(\text{dom}(f)) \rightarrow \mathbb{N}, \quad (fg)(n) = f(g(n)). \quad (2.2.1)$$

In (2.2.1) and elsewhere, we shall omit the composition sign \circ , except when strictly necessary to avoid confusion. The monoid Emb is *pointed*, i.e. it has a zero element, namely, the empty function $\emptyset \rightarrow \mathbb{N}$. The antipode map $^\dagger : \text{Emb} \rightarrow \text{Emb}$ is defined by

$$\text{dom}(f^\dagger) = \text{ran}(f), \quad f^\dagger(n) = f^{-1}(n).$$

If $A \subset \mathbb{N}$, we write P_A for the inclusion $A \hookrightarrow \mathbb{N}$. It is easily checked that

$$f^\dagger f = P_{\text{dom}f}, \quad f f^\dagger = P_{\text{ran}f}, \quad (2.2.2)$$

for any $f \in \text{Emb}$. Observe that f^\dagger is characterized as the unique element of Emb which satisfies simultaneously

$$f f^\dagger f = f \quad \text{and} \quad f^\dagger f f^\dagger = f^\dagger.$$

Thus the monoid Emb together with its antipode is a pointed *inverse monoid* that is, a pointed *inverse semigroup* with identity element. Note that Emb is the object usually denoted $\mathcal{I}(\mathbb{N})$ in the literature on semigroups (see [16, Def. 4.2], for instance).

If \mathbb{V} is a bornological vector space, the monoid Emb acts on $\ell^\infty(\mathbb{V})$ via:

$$(f_*(\alpha))_n = \begin{cases} \alpha_{f^\dagger(n)} & \text{if } n \in \text{ran}(f) \\ 0 & \text{otherwise.} \end{cases} \quad (2.2.3)$$

The subspaces $c_0(\mathbb{V})$ and $\ell^p(\mathbb{V})$ defined in 2.1.1 are *symmetric*, i.e. they are invariant under the action of Emb. Indeed, this follows from the fact that c_0 and ℓ^p are symmetric, and that if D is a bounded disk and the image

of α is contained in \mathbb{V}_D , then the following sequences of real numbers are identical

$$\|f_*(\alpha)\|_D = f_*(\|\alpha\|_D).$$

More generally, if $S \subset \ell^\infty$ is any symmetric subspace, then

$$S(\mathbb{V}) := \{\alpha \in \ell^\infty(\mathbb{V}) : (\exists D) \alpha(\mathbb{N}) \subset \mathbb{V}_D \text{ and } \|\alpha\|_D \in S\}$$

is symmetric. We denote by U the representation of Emb by partial isometries on $\ell^2(\mathbb{V})$:

$$U_f(\xi)_m = \begin{cases} \xi_n & \text{if } f(n) = m \\ 0 & \text{if } m \notin \text{ran}(f) \end{cases} \quad (\xi \in \ell^2(\mathbb{V})). \quad (2.2.4)$$

Straightforward computations show that

$$U_{fg} = U_f U_g. \quad (2.2.5)$$

Observe that U_f is a partial isometry whose initial and final space are, respectively, the closed subspaces =

$$\text{span}\{v : \text{supp}(v) \subset \text{dom}(f)\} \text{ and } \text{span}\{v : \text{supp}(v) \subset \text{ran}(f)\}.$$

This follows from (2.2.2), (2.2.5), and from the fact that if $A \subset \mathbb{N}$, then

$$U_{P_A}(v)_n = \begin{cases} v_n & \text{if } n \in A \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2.6. We will often work with sequences indexed by infinite countable sets other than \mathbb{N} . A bijection $u : \mathbb{N} \rightarrow X$ gives rise to a bounded isomorphism $\alpha \mapsto \alpha u$ between the bornological vector space $\ell^\infty(X, \mathbb{V})$ of bounded maps from X to the bornological space \mathbb{V} and the space $\ell^\infty(\mathbb{V}) = \ell^\infty(\mathbb{N}, \mathbb{V})$. If $S \subset \ell^\infty$ is a symmetric subspace, we define $S(X, \mathbb{V}) = \{su^{-1} : s \in S(\mathbb{V})\}$. Because S is symmetric by assumption, this definition does not depend on the choice of u .

Notation 2.2.7. Let $S \subset \ell^\infty$ be a symmetric subspace, X an infinite countable set and \mathbb{V} a bornological vector space. We use the following abbreviated notation: $S = S(\mathbb{N}, \mathbb{C})$, $S(X) = S(X, \mathbb{C})$ and $S(\mathbb{V}) = S(\mathbb{N}, \mathbb{V})$.

3. THE ALGEBRAS $\Gamma^\infty(\mathfrak{A})$ AND $\Gamma(R)$

Throughout this section, \mathfrak{A} will be a fixed bornological algebra, which, except in Definition 3.16, will be assumed unital. It follows straightforwardly from equations (2.1.3), (2.2.3), and (2.2.4) that

$$\text{diag}(f_*(\alpha))U_f = U_f \text{diag}(\alpha) \quad \text{and} \quad U_f \text{diag}(\alpha)U_{f^\dagger} = \text{diag}(f_*(\alpha)), \quad (3.1)$$

where $\alpha \in \ell^\infty(\mathfrak{A})$ and $f \in \text{Emb}$. Set

$$\Gamma^\infty(\mathfrak{A}) = \text{span}\{\text{diag}(\alpha)U_f : \alpha \in \ell^\infty(\mathfrak{A}), f \in \text{Emb}\}. \quad (3.2)$$

Notice that, by equations (2.2.5) and (3.1), $\Gamma^\infty(\mathfrak{A})$ is a subalgebra of the algebra $\mathcal{B}(\ell^2(\mathfrak{A}))$. For each symmetric ideal $S \triangleleft \ell^\infty$, we write $I_{S(\mathfrak{A})}$ for the

ideal of $\Gamma^\infty(\mathfrak{A})$ generated by $\text{diag}(S(\mathfrak{A}))$. Because S is invariant under the action of Emb , then by equations (3.1) we have

$$I_{S(\mathfrak{A})} = \text{span}\{\text{diag}(\alpha)U_f : \alpha \in S(\mathfrak{A}), f \in \text{Emb}\}. \quad (3.3)$$

Note that $\Gamma^\infty(\mathfrak{A}) = I_{\ell^\infty(\mathfrak{A})}$. If X is any infinite countable set, we may also consider the subalgebra $\Gamma^\infty(X, \mathfrak{A}) \subset \mathcal{B}(\ell^2(X, \mathfrak{A}))$ spanned by $\text{diag}(\ell^\infty(X, \mathfrak{A}))$ and $U_{\text{Emb}(X)}$. Thus $\Gamma^\infty(\mathfrak{A}) = \Gamma^\infty(\mathbb{N}, \mathfrak{A})$. In keeping with our notational conventions 2.1.2 and 2.2.7, we write $\Gamma^\infty = \Gamma^\infty(\mathbb{C})$ and $\Gamma^\infty(X) = \Gamma^\infty(X, \mathbb{C})$.

Notation 3.4. Since \mathfrak{A} is assumed to be unital, every sequence $a = \{a_n\}$ in $\ell^2(\mathfrak{A})$ can be written uniquely as $a = \sum_n a_n e_n$, where $e_n \in \ell^2(\mathfrak{A})$ is defined by $(e_n)_i = \delta_{n,i}$. Notice that the elements of $\Gamma^\infty(\mathfrak{A})$ are \mathfrak{A} -linear operators on the right \mathfrak{A} -module $\ell^2(\mathfrak{A})$. As usual, we identify an \mathfrak{A} -linear operator $A \in \mathcal{B}(\ell^2(\mathfrak{A}))$ with the infinite matrix $(A_{ij})_{i,j \in \mathbb{N}}$ with entries in \mathfrak{A} defined by

$$Ae_n = \sum_k A_{kn} e_k.$$

We denote by E_{ij} the matrix $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Given a matrix $A = (A_{ij})_{i,j \in \mathbb{N}}$ with entries in \mathfrak{A} , and $i, j \in \mathbb{N}$, we set:

$$\begin{aligned} J_i(A) &= \{j : A_{ij} \neq 0\}, I_j(A) = \{i : A_{ij} \neq 0\}, \\ r_i(A) &= \#J_i(A), c_j(A) := \#I_j(A), \\ r(A) &:= \max_i r_i(A), \quad c(A) := \max_i c_i(A), \\ N(A) &:= \max\{r(A), c(A)\}, \end{aligned}$$

where $r_i(A), c_j(A), N(A) \in \mathbb{N} \cup \{\infty\}$. If R is a ring, we write $\Gamma(R)$ for *Karoubi's cone*

$$\Gamma(R) = \{A \in R^{\mathbb{N} \times \mathbb{N}} : N(A) < \infty \text{ and } \{A_{i,j} : i, j \in \mathbb{N}\} \text{ is finite}\}. \quad (3.5)$$

It was shown in ([9, Lemma 4.7.1]) that $\Gamma(R)$ is isomorphic to $R \otimes \Gamma(\mathbb{Z})$, for any ring R . Since all the rings we shall be considering in this article are \mathbb{Q} -algebras, we shall write

$$\Gamma = \Gamma(\mathbb{Q}).$$

Observe that definition (3.5) extends to matrices indexed by any countable infinite set X ; if $f : \mathbb{N} \rightarrow X$ is a bijection, $\Gamma(X, R) \subset R^{X \times X}$ is the image of $\Gamma(R)$ under the map $A \mapsto U_f A U_{f^{-1}}$. Thus $\Gamma(R) = \Gamma(\mathbb{N}, R)$; we shall write $\Gamma(X) = \Gamma(X, \mathbb{Q})$.

The following lemmas will be useful in obtaining characterizations of $\Gamma^\infty(\mathfrak{A})$, $I_{S(\mathfrak{A})}$ and $\Gamma(R)$ as rings of matrices acting on $\ell^2(\mathfrak{A})$ and $R^{(\mathbb{N})}$, respectively. If $A \in R^{\mathbb{N} \times \mathbb{N}}$ is such that $N(A) < \infty$, we write $\Gamma(R)A\Gamma(R)$ to denote the set

$$\Gamma(R)A\Gamma(R) := \left\{ \sum_{j=1}^n P_j A Q_j : P_j, Q_j \in \Gamma(R) \text{ for all } j = 1, \dots, n \text{ and } n \in \mathbb{N} \right\}.$$

Lemma 3.6. *Let R be a unital ring, $A = (A_{ij})_{i,j \in \mathbb{N}} \in R^{\mathbb{N} \times \mathbb{N}}$ a matrix such that $N(A) < \infty$ and $r(A) > 1$. Then*

- (1) *$A = A_1 + A_2 + \cdots + A_k$, where $A_i \in \Gamma(R)A\Gamma(R)$, $r(A_i) < r(A)$ and $c(A_i) \leq c(A)$ for all $i = 1, \dots, k$.*
- (2) *If in addition R is a unital bornological algebra and $S \triangleleft \ell^\infty$ is a symmetric ideal such that $\{A_{ij}\} \in S(\mathbb{N} \times \mathbb{N}, R)$, then $\{(A_l)_{ij}\} \in S(\mathbb{N} \times \mathbb{N}, R)$, for all $l = 1, \dots, k$.*

Proof. (1) We first establish some notation and make some reductions. Let

$$r = r(A)$$

$$I = \{i \in \mathbb{N} : \text{the } i^{\text{th}} \text{ row of } A \text{ has } r \text{ nonzero entries}\}.$$

For $i \in I$, let

$$h_i(1) < h_i(2) < \cdots < h_i(r)$$

be the columns where the nonzero entries of row i occur. Let A_r denote the matrix obtained from A upon multiplying by zero those rows that have less than r nonzero entries. Then $A_r \in \Gamma(R)A\Gamma(R)$, and

$$r(A_r) = r, \quad r(A - A_r) < r, \quad c(A_r) \leq c(A), \quad \text{and } c(A - A_r) \leq c(A).$$

Thus it suffices to prove (1) for A_r . Hence we may assume that $A = A_r$, that is, that all nonzero rows of A have exactly r nonzero entries. Furthermore, since there are at most $c(A)$ nonzero entries in each column of A , the set I can be written as a disjoint union $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_s$ with $s \leq c(A)$ and such that each I_t ($1 \leq t \leq s$) satisfies the following property:

$$i \neq j \in I_t \Rightarrow h_i(1) \neq h_j(1).$$

Proceeding as above we see that we may assume that $s = 1$. Notice that if A' is obtained from A by permuting its rows, then $A' = U_f A$ for some bijection $f : \mathbb{N} \rightarrow \mathbb{N}$. Therefore, $\Gamma(R)A\Gamma(R) = \Gamma(R)A'T(R)$, $r(A') = r(A)$, and $c(A') = c(A)$, so we may assume that $A = A'$. Thus we will assume that the rows of A are ordered so that if $i, j \in I$, then $h_i(1) < h_j(1)$ if and only if $i < j$.

Thus, it only remains to show (1) for matrices A such that for I and h_i as above:

$$\text{a) All nonzero rows of } A \text{ have exactly } r \text{ nonzero entries.} \quad (3.7)$$

$$\text{b) } i < j \iff h_i(1) < h_j(1) \text{ for all } i, j \in I. \quad (3.8)$$

We shall proceed by induction on

$$M_A = \max_{j \in I} \#\{i \in I : A_{ih_j(1)} \neq 0\}.$$

Notice that the right-hand side of the equation above is bounded by $c(A)$, so $M_A \in \mathbb{N}$. First assume that $M_A = 1$. Then for all $i, j \in I$ we have that $A_{ih_j(1)} \neq 0$ if and only if $i = j$. Set

$$A_1 = \sum_{i \in I} A_{ih_i(1)} E_{ih_i(1)} = \left(\sum_{i \in I} E_{ii} \right) A \left(\sum_{j \in I} E_{h_j(1)h_j(1)} \right) \in \Gamma(R)A\Gamma(R).$$

Then

$$r(A_1) < r, \quad r(A - A_1) < r, \quad c(A_1) \leq c(A), \quad \text{and} \quad c(A - A_1) \leq c(A),$$

so the statement in (1) holds for A . Assume now that $M_A > 1$ and that (1) holds for matrices B satisfying 3.7 and 3.8, and such that $M_B < M_A$. Let

$$i_1 := \min I, \quad K_1 := \{j \in I : A_{i_1 h_j(1)} \neq 0\}.$$

For $n \geq 1$ such that $\bigcup_{j=1}^{n-1} K_j \neq I$, let

$$i_n := \min I \setminus \bigcup_{j=1}^{n-1} K_j, \quad \text{and} \quad K_n := \{j \in I \setminus \bigcup_{l=1}^{n-1} K_l : A_{i_n h_j(1)} \neq 0\}.$$

Let

$$\mathcal{J} = \begin{cases} \{1, 2, \dots, n\}, & \text{if } \bigcup_{j=1}^n K_j = I. \\ \mathbb{N}, & \text{otherwise.} \end{cases}$$

We claim that

$$\text{a) } i_n > i_{n-1} \quad \forall n \in \mathcal{J} \quad \text{and} \quad \text{b) } I = \bigcup_{j \in \mathcal{J}} K_j. \quad (3.9)$$

In fact a) follows from the inequality

$$i_n = \min I \setminus \bigcup_1^{n-1} K_j \geq \min I \setminus \bigcup_1^{n-2} K_j = i_{n-1}$$

and the fact that $i_n \neq i_{n-1}$ because $i_n \notin K_{n-1}$ and $i_{n-1} \in K_{n-1}$. It is clear that b) holds when \mathcal{J} is finite. Assume now that \mathcal{J} infinite. If $k \in I$, then either $k \in \{i_n : n \in \mathcal{J}\} \subset \bigcup K_j$ or, by a), there exists $n \in \mathcal{J}$ such that

$$k < i_n = \min I \setminus \bigcup_1^{n-1} K_j.$$

This implies that $k \in \bigcup_1^{n-1} K_j$. Thus b) holds also when \mathcal{J} is infinite, and both claims are proven. Now set

$$B := \sum_{n \in \mathcal{J}, j \in \mathbb{N}} A_{i_n j} E_{i_n j} = \left(\sum_{n \in \mathcal{J}} E_{i_n i_n} \right) A \in \Gamma(R) A \Gamma(R).$$

Notice that B is obtained from A by multiplying by zero the i^{th} row whenever $i \notin \{i_n : n \in \mathcal{J}\}$. Therefore B satisfies 3.7 and 3.8, $r(B) = r$, and $c(B) \leq c(A)$. We next show that $M_B = 1$. We begin by noting that $B_{i_m i_n(1)} \neq 0$ implies that $A_{i_m i_n(1)} \neq 0$. Then $i_n(1) \geq i_m(1)$, which implies by 3.8 that $i_n \geq i_m$, which in turn implies, by part a) of equation (3.9), that $n \geq m$. Now, if $n > m$ we would have

$$i_n \notin \bigcup_1^{n-1} K_j \supseteq \bigcup_1^m K_j.$$

Then $i_n \notin K_m$ and $i_n \notin \bigcup_1^{m-1} K_j$, which implies that $A_{i_m i_n(1)} = 0$, a contradiction. Thus $n = m$ and $M_B = 1$, as claimed. Set $C = A - B$; we have $r(C) = r$ and $c(C) \leq c(A)$. Notice that C is obtained from A upon multiplying by zero the i_n^{th} row for all $n \in \mathcal{J}$. Besides, the i^{th} row of C is nonzero if and only if $i \in I_C := I \setminus \{i_n : n \in \mathcal{J}\}$, and in that case it is equal to the i^{th} row of A . Therefore, C satisfies 3.7 and 3.8. We next prove that $M_C < M_A$, which will conclude the proof of part (1). If $i, j \in I_C$, then $A_{i h_j(1)} = 0$ implies that $C_{i h_j(1)} = 0$. On the other hand, by part b) of equation (3.9), we can choose $n \in \mathcal{J}$ such that $j \in K_n$. Then $A_{i_n h_j(1)} \neq 0$, whereas $C_{i_n h_j(1)} = 0$. It follows that $M_C \leq M_A - 1$. This concludes the proof of part (1). Part (2) holds because for $l = 1, \dots, k$, $\{(A_l)_{ij}\}$ is obtained upon multiplication of $\{A_{ij}\}$ by bounded sequences and by permutations of terms. \square

Lemma 3.10. *Let $A = (A_{ij})_{i,j \in \mathbb{N}}$ be a matrix with entries in a unital ring R such that $N(A) < \infty$. Then*

- (1) $A = A_1 + A_2 + \dots + A_k$, where $A_i \in \Gamma(R)A\Gamma(R)$, and $N(A_i) \leq 1$, for all $i = 1, \dots, k$.
- (2) If in addition R is a bornological algebra and $S \triangleleft \ell^\infty$ is a symmetric ideal such that $\{A_{ij}\} \in S(\mathbb{N} \times \mathbb{N}, R)$, then $\{(A_l)_{ij}\} \in S(\mathbb{N} \times \mathbb{N}, R)$, for all $l = 1, \dots, k$.

Proof. Use Lemma 3.6 and proceed by induction on $r(A)$ to write

$$A = \sum_1^k B_i, \quad \text{where } r(B_i) = 1, \quad c(B_i) \leq c(A), \quad \text{and } B_i \in \Gamma(R)A\Gamma(R).$$

Next apply the same procedure to each transpose matrix B_i^t to get the decomposition in (1). The second statement follows from the second part of Lemma 3.6. \square

Proposition 3.11. *Let $A = (A_{ij})_{i,j \in \mathbb{N}}$ be a matrix with entries in a ring R . Then $N(A) \leq 1$ if and only if $A = \text{diag}(\alpha)U_f$, where $f \in \text{Emb}$ and $\alpha \in R^\mathbb{N}$ are defined as follows:*

$$f(j) = i \iff A_{ij} \neq 0 \quad \alpha(i) = \begin{cases} A_{ij}, & \text{if } i = f(j) \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For f and α as in the proposition, the n^{th} column of A is

$$\begin{aligned} (\text{diag}(\alpha)U_f)(e_n) &= \begin{cases} \alpha(n)e_{f(n)}, & \text{if } n \in \text{dom}(f) \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} A_{f(n)n}e_{f(n)}, & \text{if } n \in \text{dom}(f) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

\square

The following proposition is well known. We include a proof since we have not been able to find one in the literature.

Proposition 3.12. *Let R be a ring. Then the set*

$$\{\text{diag}(\alpha)U_f : f \in \text{Emb and } \{\alpha_n : n \in \mathbb{N}\} \subset R \text{ finite} \}$$

generates $\Gamma(R)$ as an abelian group.

Proof. Let $X_R \subset \Gamma(R)$ be the set of elements of the proposition. By [9, Lemma 4.7.1], the map $\phi : \Gamma\mathbb{Z} \otimes R \rightarrow \Gamma(R)$, $\phi(A \otimes x)_{i,j} = A_{i,j}x$ is an isomorphism. This map sends the subgroup generated by the elements $x \otimes r$ ($x \in X_{\mathbb{Z}}$, $r \in R$) onto the subgroup generated by X_R . Thus it suffices to prove the Proposition for $R = \mathbb{Z}$. Let $A \in \Gamma(\mathbb{Z})$. By Lemma 3.10 and Proposition 3.11, $A = \sum_k A_k$, where $\Gamma \ni A_k = \text{diag}(\alpha^k)U_{f_k}$, for some \mathbb{Z} -valued sequence α^k . Besides, $A_k \in \Gamma(\mathbb{Z})$ implies that the set $\{\alpha_n^k : n \in \mathbb{N}\}$ is finite. \square

Corollary 3.13. *Let \mathfrak{A} be a unital bornological algebra. Then Karoubi's cone $\Gamma(\mathfrak{A})$ is a subalgebra of $\Gamma^\infty(\mathfrak{A})$.*

Proposition 3.14. *Let \mathfrak{A} be a unital bornological algebra, $S \triangleleft \ell^\infty$ a symmetric ideal, and $I_{S(\mathfrak{A})} \triangleleft \Gamma^\infty(\mathfrak{A})$ the ideal defined in equation (3.3). Then*

$$I_{S(\mathfrak{A})} = \{A = (A_{ij})_{i,j \in \mathbb{N}} : \{A_{ij}\} \in S(\mathbb{N} \times \mathbb{N}) \text{ and } N(A) < \infty\}. \quad (3.15)$$

Proof. Let D_S denote the set on the right hand side of equation (3.15). By Lemma 3.10 and Proposition 3.11, a matrix A belongs to D_S if and only if $A = \sum A_k$, with $A_k = \text{diag}(\alpha_k)U_{f_k} \in D_S$. Further, we may choose α_k and f_k such that $\text{supp}(\alpha_k) = \text{ran}(f_k)$. Under these conditions, $A_k \in D_S$ if and only if $\alpha^k \in S$. This shows that $A \in D_S$ if and only if $A \in I_S$. \square

Definition 3.16. If \mathfrak{A} is a not necessarily unital bornological algebra, and $S \triangleleft \ell^\infty$ is a symmetric ideal, $I_{S(\mathfrak{A})}$ is defined by (3.15).

Example 3.17. Let

$$c_f = \{\alpha \in \ell^\infty : \text{supp}(\alpha) \text{ is finite} \}.$$

Then

$$I_{c_f(\mathfrak{A})} = M_\infty(\mathfrak{A}) := \{A : \exists n \in \mathbb{N} \text{ such that } A_{ij} = 0 \text{ if either } i > n \text{ or } j > n \}.$$

We shall write $M_\infty = M_\infty\mathbb{Q}$.

Remark 3.18. Let \mathfrak{A} be a unital bornological algebra, $I \triangleleft \Gamma^\infty(\mathfrak{A})$ a two-sided ideal and $T \in I$. Then by Lemma 3.10 and Remark 3.11, we can write

$$T = \sum_{i=1}^n \text{diag}(\alpha^i)U_{f_i} \text{ with } \text{diag}(\alpha^i)U_{f_i} \in I, \quad (3.19)$$

where $f_i \in \text{Emb}$ and $\alpha^i \in \ell^\infty(\mathfrak{A})$. Similarly, if R is a unital ring and $T \in I \triangleleft \Gamma(R)$, then we can also write T as in (3.19) but now with α^i such that the set $\{\alpha_n^i : n \in \mathbb{N}\} \subset R$ is finite.

4. THE TWO-SIDED IDEALS OF $\Gamma^\infty(\mathbb{C})$ AND THOSE OF $\mathcal{B}(\ell^2(\mathbb{N}))$

Calkin's theorem in [2, Theorem 1.6]), as restated by Garling in [17, Theorem 1], establishes a bijective correspondence between the set of proper two-sided ideals of $\mathcal{B} = \mathcal{B}(\ell^2)$ and the set of proper symmetric ideals of ℓ^∞ . Calkin defined this correspondence in terms of the sequence of singular values of a compact operator. It can also be described as follows: an ideal $J \triangleleft \mathcal{B}$ is mapped to the symmetric ideal

$$S(J) = \{\alpha \in \ell^\infty : \text{diag}(\alpha) \in J\}. \quad (4.1)$$

The inverse correspondence maps a symmetric ideal S in ℓ^∞ to the two-sided ideal

$$\mathcal{B} \triangleright J_S = \langle \text{diag}(\alpha) : \alpha \in S \rangle \quad (4.2)$$

We refer the reader to [25, Theorem 2.5] for further details. Recall that, by another result of Calkin [2, Theorem 1.4], the Calkin algebra \mathcal{B}/\mathcal{K} is simple. On the other hand, it is easily checked that $c_0 \triangleleft \ell^\infty$ is maximal among proper symmetric ideals. Thus, by mapping ℓ^∞ to \mathcal{B} we extend the correspondence above to a bijection between the family of symmetric ideals of ℓ^∞ and that of two-sided ideals of \mathcal{B} . In Theorem 4.5 below we show that Calkin's correspondence carries over to ideals in Γ^∞ . We will make use of the following lemma.

Lemma 4.3. *Let $\alpha \in \ell^\infty$, $f \in \text{Emb}$ and let $I \triangleleft \Gamma^\infty$ a two-sided ideal. Consider the operator*

$$T = \text{diag}(\alpha)U_f.$$

Then

$$T \in I \iff |T| \in I.$$

Proof. We have

$$T^*T = U_f^* \text{diag}(|\alpha|^2) U_f = \text{diag}(f_*^\dagger(|\alpha|^2)) = \text{diag}(|f_*^\dagger(\alpha)|^2).$$

Therefore, $|T| = \text{diag}(|f_*^\dagger(\alpha)|)$, and the polar decomposition of T is $T = V|T|$, where

$$V = \text{diag}(\nu_\alpha)U_f,$$

for

$$\nu_\alpha(n) = \begin{cases} 0, & \text{if } \alpha(n) = 0 \\ \frac{\alpha(n)}{|\alpha(n)|}, & \text{otherwise.} \end{cases} \quad (4.4)$$

It is now clear that $V \in \Gamma^\infty$. Thus $T \in I$ if and only if $|T| \in I$, since Γ^∞ is a $*$ -algebra and $|T| = V^*T$. \square

Theorem 4.5.

- i) *The map $S \mapsto I_S$ is a bijection between the set of symmetric ideals of ℓ^∞ and the set of two-sided ideals of Γ^∞ . Its inverse maps an ideal $I \triangleleft \Gamma^\infty$ to the symmetric ideal $S(I)$ defined as in (4.1).*

- ii) The map $J \mapsto J \cap \Gamma^\infty$ is a bijection between the sets of two-sided ideals of \mathcal{B} and those of Γ^∞ . Its inverse maps an ideal $I \triangleleft \Gamma^\infty$ to the two-sided ideal of \mathcal{B} it generates.
- iii) If $S \triangleleft \ell^\infty$ is a symmetric ideal, then $J_S \cap \Gamma^\infty = I_S$.

Proof. Let $I \triangleleft \Gamma^\infty$; write $S = S(I)$. It is clear that $I_S \subseteq I$. On the other hand, if $T = \text{diag}(\alpha)U_f \in I$, for some $\alpha \in \ell^\infty$ and $f \in \text{Emb}$, then, by Lemma 4.3,

$$\text{diag}(f_*^\dagger(|\alpha|)) = |T| \in I_S.$$

Hence $T \in I_S$, again by Lemma 4.3. In view of Remark 3.18, this implies that $I = I_S$. We have shown that $I_{S(I)} = I$. Let now $S \triangleleft \ell^\infty$ be a symmetric ideal. Then

$$S \subset S(I_S) \subset S(J_S) \subset S,$$

the last inclusion being due to Calkin's theorem. It follows that $S = S(I_S)$, completing the proof of part i). Next, since the ideal $\langle I_S \rangle \triangleleft \mathcal{B}(\ell^2)$ generated by I_S is also generated by $\text{diag}(S)$ we have $\langle I_S \rangle = J_S$, by Calkin's theorem. Now, again by Calkin's theorem,

$$S \subset S(J_S \cap \Gamma^\infty) \subset S(J_S) = S.$$

Thus $J_S \cap \Gamma^\infty = I_S$, by part i). We have proven part iii) and also shown that $\langle I_S \rangle \cap \Gamma^\infty = I_S$. Moreover, by parts i) and iii) we have

$$\text{diag}(\ell^\infty) \cap J_S = \text{diag}(\ell^\infty) \cap J_S \cap \Gamma^\infty = \text{diag}(\ell^\infty) \cap I_S = \text{diag}(S).$$

It follows that $\langle J_S \cap \Gamma^\infty \rangle = J_S$, which ends the proof. \square

It follows from Proposition 3.14, Example 3.17 and Theorem 4.5 that

$$I \cap \Gamma(\mathbb{C}) = M_\infty(\mathbb{C})$$

for every proper ideal $I \triangleleft \Gamma^\infty$. The next proposition shows that in fact $M_\infty(\mathbb{C})$ is the only proper ideal of $\Gamma(\mathbb{C})$.

Proposition 4.6. *Let k be a field. Then $M_\infty(k)$ is the only proper two-sided ideal of $\Gamma(k)$.*

Proof. It is well known and easy to check that $M_\infty(R) \triangleleft \Gamma(R)$ for any ring R . Let $I \neq 0$ be a two-sided ideal of $\Gamma(k)$, and let $A \neq 0$, $A \in I$. If i_0 and j_0 are such that $A_{i_0 j_0} \neq 0$, then

$$E_{ij} = (A_{i_0 j_0})^{-1} E_{i i_0} A E_{j_0 j} \in I \quad \forall i, j \quad (4.7)$$

This shows that $M_\infty(k) \subseteq I$. Assume that the inclusion is strict. Let $A \in I \setminus M_\infty(k)$. By Remark (3.18), we may assume that $A = \text{diag}(\alpha)U_f$ for $f \in \text{Emb}$ and $\alpha \in k^\mathbb{N}$, where $\text{Im}(\alpha) = \{\alpha_n : n \in \mathbb{N}\}$ is finite and $\text{supp}(\alpha) = \text{dom} f \subset \mathbb{N}$ is infinite. Because k is a field, we can multiply A on the left by a diagonal matrix in $\Gamma(k)$ to conclude that $U_f \in I$. But since $\text{ran}(f)$ is infinite, there are bijections $g : \mathbb{N} \rightarrow \text{dom}(f)$ and $h : \text{ran}(f) \rightarrow \mathbb{N}$ such that $hfg = 1$. Hence I must contain $1 = U_h U_f U_g$. \square

5. THE ALGEBRA $\Gamma^\infty(\mathfrak{A})$ AS A CROSSED PRODUCT

Let $2^\mathbb{N}$ denote the submonoid of idempotent elements of Emb

$$2^\mathbb{N} = \{p : p \in \text{Emb} \text{ } p^2 = p\} \subset \text{Emb}.$$

Note that if $p \in 2^\mathbb{N}$, then for $A = \text{ran}(p) = \text{dom}(p)$, we have $U_p = \text{diag}(\chi_A)$, the diagonal matrix on the sequence

$$(\chi_A)_n = \begin{cases} 1 & n \in A \\ 0 & n \notin A. \end{cases}$$

We will often identify p , $U_p = \text{diag}(\chi_A)$, and χ_A . Notice that

$$f_*(p)f = fp. \quad (5.1)$$

The subspace of Γ generated by the image of $2^\mathbb{N}$ under $f \mapsto U_f$ is the subalgebra

$$\mathcal{P} = \text{span}\{U_p : p \in 2^\mathbb{N}\} \subset \Gamma.$$

We also consider the monoid algebras $\mathbb{Q}[2^\mathbb{N}]$ and $\mathbb{Q}[\text{Emb}]$, and the two-sided ideals

$$I = \langle \{\chi_{A \sqcup B} - \chi_A - \chi_B : A, B \subset \mathbb{N}, \ A \cap B = \emptyset\} \rangle \triangleleft \mathbb{Q}[2^\mathbb{N}], \quad (5.2)$$

$$J = \langle \{\chi_{A \sqcup B} - \chi_A - \chi_B : A, B \subset \mathbb{N}, \ A \cap B = \emptyset\} \rangle \triangleleft \mathbb{Q}[\text{Emb}]. \quad (5.3)$$

Observe that I and J contain the element

$$\chi_{A \sqcup B} - \chi_A - \chi_B - \chi_{A \cap B}$$

for any pair of not necessarily disjoint subsets $A, B \subset \mathbb{N}$.

Lemma 5.4.

i) $\mathcal{P} = \mathbb{Q}[2^\mathbb{N}]/I$.

ii) $\Gamma = \mathbb{Q}[\text{Emb}]/J$

iii) If \mathfrak{A} is a unital bornological algebra, then $\ell^\infty(\mathfrak{A}) \otimes_{\mathcal{P}} \Gamma \cong \Gamma^\infty(\mathfrak{A})$ as \mathcal{P} -bimodules.

Proof. It is clear that there are natural surjective algebra maps

$$\pi_1 : \mathbb{Q}[2^\mathbb{N}]/I \rightarrow \mathcal{P} \text{ and}$$

$$\pi_2 : \mathbb{Q}[\text{Emb}]/J \rightarrow \Gamma,$$

and a natural surjective \mathcal{P} -bimodule homomorphism

$$\pi_3 : \ell^\infty \otimes_{\mathcal{P}} \Gamma \rightarrow \Gamma^\infty.$$

Let $\xi = \sum_{j=1}^n \lambda_j \chi_{A_j} \in \mathbb{Q}[2^\mathbb{N}]$ represent an element $\in \ker \pi_1$; for each subset $F \subset \{1, \dots, n\}$, let $A_F = \bigcap_{j \in F} A_j \cap \bigcap_{j \notin F} A_j^c$. From $\pi_1(\xi)|_{A_F} = 0$ we get

$$A_F \neq \emptyset \Rightarrow \sum_{j \in F} \lambda_j = 0.$$

Next note that $\bigcup_{i=1}^n A_i = \sqcup_F A_F$; hence, modulo I , we have

$$\begin{aligned}\xi &\equiv \sum_F \sum_{j=1}^n \lambda_j \chi_{A_j \cap A_F} \\ &= \sum_F \left(\sum_{j \in F} \lambda_j \right) \chi_{A_F} = 0.\end{aligned}$$

This proves i). In order to prove ii) we have to show that $\ker(\pi_2) = 0$. Let $\xi = \sum_{j=1}^n \lambda_j f_j \in \mathbb{Q}[\text{Emb}]$ be a representative of an element in $\ker(\pi_2)$. Let $A_i = \text{dom} f_i$, and let A_F be as above; then $\xi \equiv \sum_F \xi \chi_{A_F}$. Hence we may assume that the A_i are disjoint. Furthermore, upon replacing ξ by $\xi \chi_{A_i}$, and eliminating zero elements of Emb , we may assume that $A_1 = \dots = A_n$. For each $j \in \mathbb{N}$, we have

$$\sum_{i=1}^n \lambda_i e_{f_i(j)} = 0. \quad (5.5)$$

Let $K = \{f_i(j) : i = 1, \dots, n\}$; for each $k \in K$, let $D_k = \{i : f_i(j) = k\}$. Then $D(j) := \{D_k\}_{k \in K}$ is a partition of $\{1, \dots, n\}$, and $\sum_{i \in D_k} \lambda_i = 0$. There is a finite set \mathcal{D} of partitions arising in this way, since the number of all partitions of $\{1, \dots, n\}$ is finite. For each $D \in \mathcal{D}$, let $J_D = \{j \in \mathbb{N} : D(j) = D\}$. Then $\sqcup_{D \in \mathcal{D}} J_D = \mathbb{N}$, and $\xi \equiv \sum_D \xi \cdot \chi_D$. Hence, upon replacing ξ by $\xi \chi_D$ if necessary, we may assume that \mathcal{D} has only one element $D = \{D_1, \dots, D_r\}$. But $\xi \equiv \sum_i \chi_{D_i} \xi$, so we further reduce to the case when $r = 1$. This means that $f_1 = \dots = f_n$ and, by (5.5), $\sum_i \lambda_i f_i$ is the zero element of $\mathbb{Q}[\text{Emb}]$. We have proved ii). To prove iii) we must show that π_3 is injective. Let $\xi = \sum_{i=1}^n \alpha^{(i)} \otimes U_{f_i} \in \ker \pi_3$. Because

$$\alpha \otimes U_f = \alpha \chi_{\text{supp}(\alpha) \cap \text{ran} f} \otimes \chi_{\text{supp}(\alpha) \cap \text{ran} f} U_f \in \ell^\infty(\mathfrak{A}) \otimes_{\mathcal{P}} \Gamma,$$

we may assume that $\text{supp}(\alpha_i) = \text{ran}(f_i)$ ($i = 1, \dots, n$). Proceeding as above, we may assume that $\text{dom} f_1 = \dots = \text{dom} f_n$. For each $j \in \mathbb{N}$, we have

$$\sum_{i=1}^n \alpha_j^{(i)} e_{f_i(j)} = 0. \quad (5.6)$$

Proceeding as above again, we may reduce to the case $f_1 = \dots = f_n$. By (5.6), we have $\sum_{i=1}^n \alpha^{(i)} = 0$. Thus

$$\xi = \sum_{i=1}^n \alpha^{(i)} \otimes U_{f_i} = \left(\sum_{i=1}^n \alpha^{(i)} \right) \otimes U_{f_1} = 0.$$

□

Remark 5.7. Given any monoid M , a \mathbb{Q} -vector space representation of M is the same thing as module over the monoid ring $\mathbb{Q}[M]$. In view of Lemma 5.4, the modules over \mathcal{P} and Γ correspond to those representations of the inverse monoids $2^{\mathbb{N}}$ and Emb which are tight in the sense of Exel (see [16, Def. 13.1 and Prop. 11.9]).

Because Emb is a monoid, if \mathcal{A} is an algebra on which Emb acts by algebra endomorphisms we can form the *crossed product* $\mathcal{A} \# \text{Emb}$. As a vector space $\mathcal{A} \# \text{Emb} = \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\text{Emb}]$ with multiplication given by

$$(a \# f)(b \# g) = af_*(b) \# fg \quad (5.8)$$

Here $\# = \otimes$ and $f_*(b)$ denotes the action of f on Emb . Now assume that the Emb -algebra \mathcal{A} is also a \mathcal{P} -algebra, that is, it is an algebra and a \mathcal{P} -bimodule, and these operations are compatible in the sense that

$$(ap)b = a(pb) \quad (a, b \in \mathcal{A}, p \in \mathcal{P}).$$

Further assume that \mathcal{A} is central as a \mathcal{P} -bimodule, i.e. $pa = ap$ ($a \in \mathcal{A}$, $p \in \mathcal{P}$), and that

$$pa = p_*(a) \quad (p \in 2^{\mathbb{N}}).$$

Under all these conditions, we say that \mathcal{A} is an *Emb-bundle* (cf. [1, Def. 2.10]). For $J \triangleleft \mathbb{Q}[\text{Emb}]$ as in (5.3), we have

$$\begin{aligned} \mathcal{A} \# \text{Emb} \triangleright \mathcal{A} \# J &= \text{span}\{r \# j : r \in \mathcal{A}, j \in J\} \text{ and} \\ \mathcal{A} \# \text{Emb} \triangleright L &= \text{span}\{rp \# h - r \# ph : r \in \mathcal{A}, p \in \mathcal{P}, h \in \text{Emb}\}. \end{aligned}$$

Set

$$\mathcal{A} \#_{\mathcal{P}} \Gamma = \mathcal{A} \# \text{Emb} / (L + \mathcal{A} \# J). \quad (5.9)$$

Thus, $\mathcal{A} \#_{\mathcal{P}} \Gamma = \mathcal{A} \otimes_{\mathcal{P}} \Gamma$ as left \mathcal{P} -modules, and the product is that induced by (5.8); we have

$$(a \# U_f)(b \# U_g) = af_*(b) \# U_{fg} \in \mathcal{A} \#_{\mathcal{P}} \Gamma. \quad (5.10)$$

Proposition 5.11. *Let \mathfrak{A} be a bornological algebra. The map*

$$\ell^\infty(\mathfrak{A}) \#_{\mathcal{P}} \Gamma \rightarrow \Gamma^\infty(\mathfrak{A}), \quad \alpha \# U_f \mapsto \text{diag}(\alpha) U_f \quad (5.12)$$

is an isomorphism of \mathcal{P} -algebras. If $S \triangleleft \ell^\infty$ is a symmetric ideal, then (5.12) sends $S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma$ isomorphically onto $I_{S(\mathfrak{A})} \triangleleft \Gamma^\infty(\mathfrak{A})$.

Proof. Assume first that \mathfrak{A} is unital. Then the map (5.12) is the same as that of Lemma 5.4(iii). Hence, it is bijective. By (3.1) and (5.10), it is an algebra homomorphism. This proves the first assertion in the unital case; the second is immediate from the fact that (5.12) is bijective and maps $S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma$ onto $I_{S(\mathfrak{A})}$. For not necessarily unital \mathfrak{A} , write $\tilde{\mathfrak{A}}$ for its unitalization as a bornological algebra. We have an exact sequence

$$0 \rightarrow S(\mathfrak{A}) \rightarrow S(\tilde{\mathfrak{A}}) \rightarrow S \rightarrow 0. \quad (5.13)$$

Observe that the inclusion $\mathbb{C} \subset \tilde{\mathfrak{A}}$ induces a \mathcal{P} -module homomorphism $S \rightarrow S(\tilde{\mathfrak{A}})$ which splits the sequence (5.13). Hence we get an exact sequence

$$0 \rightarrow S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma \rightarrow S(\tilde{\mathfrak{A}}) \#_{\mathcal{P}} \Gamma \rightarrow S \#_{\mathcal{P}} \Gamma \rightarrow 0.$$

Combining this sequence with the unital case of the proposition, we obtain an isomorphism

$$S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma \xrightarrow{\cong} \ker(I_{S(\mathfrak{A})} \rightarrow I_S) = I_{S(\mathfrak{A})}.$$

□

6. MORE PROPERTIES OF \mathcal{P} , Γ^∞ AND Γ

6.1. Γ^∞ as an infinite sum ring. We begin this section by recalling some definitions from [28] and [9]. A *sum ring* (R, x_0, x_1, y_0, y_1) consists of a unital ring R and elements $x_0, x_1, y_0, y_1 \in R$ satisfying:

$$\begin{aligned} y_0 x_0 &= y_1 x_1 = 1 \\ x_0 y_0 + x_1 y_1 &= 1. \end{aligned} \quad (6.1.1)$$

If R is a sum ring, the map

$$\oplus : R \times R \longrightarrow R, \text{ defined by } r \oplus s = x_0 r y_0 + x_1 s y_1, \quad (6.1.2)$$

is a unital ring homomorphism. An *infinite sum ring* consists of a sum ring R equipped with a unital ring homomorphism

$$\Phi : R \longrightarrow R \text{ such that } r \oplus \Phi(r) = \Phi(r). \quad (6.1.3)$$

The concept of infinite sum ring was introduced by Wagoner in [28]. He showed that if R is unital, then the following is an infinite sum ring:

$$\Gamma^W(R) := \{A \in R^{\mathbb{N} \times \mathbb{N}} : A \cdot M_\infty R \subset M_\infty R \supset M_\infty R \cdot A\}.$$

We may regard $\Gamma^W(R)$ as a multiplier algebra of $M_\infty R$. One checks that a matrix $A \in \Gamma^W(R)$ if and only if every row and every column of A has finite support. Let

$$f_i : \mathbb{N} \rightarrow \mathbb{N}, \quad f_i(n) = 2n - i \quad (i = 0, 1) \quad (6.1.4)$$

The elements $x_i = U_{f_i^\dagger}$, $y_i = U_{f_i}$ satisfy conditions (6.1.1). The homomorphism Φ is defined by

$$\Phi(A) = \sum_{k=0}^{\infty} x_1^k x_0 A y_0 y_1^k = \sum_{k,i,j} A_{ij} E_{2^{k+1}i+2^k-1, 2^{k+1}j+2^k-1}. \quad (6.1.5)$$

This map is well-defined because $(k, i) \mapsto 2^{k+1}i + 2^k - 1$ is one-to-one; Wagoner showed in [28, pp 355] that it satisfies (6.1.3). Observe that the x'_i 's and y'_i 's are elements of $\Gamma(R)$. It is not hard to check, and noticed in [9, 4.8.2], that $\Phi(\Gamma(R)) \subset \Gamma(R)$, whence $\Gamma(R)$ is an infinite sum ring too. Now we remark that if \mathfrak{A} is a bornological algebra, then

$$\Gamma(\mathfrak{A}) \subset \Gamma^\infty(\mathfrak{A}) \subset \Gamma^W(\mathfrak{A}).$$

Furthermore, Φ also sends $\Gamma^\infty(\mathfrak{A})$ to itself. Thus if \mathfrak{A} is unital, then $\Gamma^\infty(\mathfrak{A})$ is an infinite sum ring. We record this in the following proposition.

Proposition 6.1.6. *Let \mathfrak{A} be a unital bornological algebra, and let f_i be as in (6.1.4) and Φ as in (6.1.5). Then $(\Gamma^\infty(\mathfrak{A}), U_{f_0^\dagger}, U_{f_1^\dagger}, U_{f_0}, U_{f_1}, \Phi)$ is an infinite sum ring.*

Corollary 6.1.7. *Let $F : \mathbb{C}\text{-Alg} \rightarrow \mathfrak{Ab}$ be a functor. Assume that the restriction of F to unital \mathbb{C} -algebras is split-exact and M_2 -stable. Then $F(\Gamma^\infty(\mathfrak{A})) = 0$ for any unital bornological algebra \mathfrak{A} . If furthermore F is split exact on all \mathbb{C} -algebras, then $F(\Gamma^\infty(\mathfrak{A})) = 0$ for any, not necessarily unital bornological algebra \mathfrak{A} .*

Proof. Immediate from Proposition 6.1.6 and [6, Proposition 2.3.1]. \square

Examples 6.1.8. Both Weibel's homotopy algebraic K -theory [30] and periodic cyclic homology [14] are M_2 -stable and excisive on all \mathbb{Q} -algebras. Hence if \mathfrak{A} is a bornological algebra, then

$$KH_*(\Gamma^\infty(\mathfrak{A})) = HP_*(\Gamma^\infty(\mathfrak{A})) = 0.$$

Algebraic K -theory groups K_n are split exact and M_2 -stable for $n \leq 0$; the same is true of Karoubi-Villamayor K -groups KV_m for $m \geq 1$ ([19, Théorème 4.5]). Hence,

$$K_n(\Gamma^\infty(\mathfrak{A})) = KV_m(\Gamma^\infty(\mathfrak{A})) = 0 \quad (n \leq 0, m \geq 1),$$

again for all \mathfrak{A} . For positive n , the groups K_n are still split exact and M_2 -stable on unital rings. The same is true of both the Hochschild and cyclic homology groups HH_n and HC_n for $n \geq 0$; moreover these groups vanish for $n \leq -1$. Hence we have

$$K_{n+1}(\Gamma^\infty(\mathfrak{A})) = HH_n(\Gamma^\infty(\mathfrak{A})) = HC_n(\Gamma^\infty(\mathfrak{A})) = 0 \quad (n \geq 0)$$

for any unital bornological algebra \mathfrak{A} .

6.2. Flat ideals of Γ^∞ and ℓ^∞ .

Proposition 6.2.1. *Every finitely generated ideal of ℓ^∞ is principal and projective.*

Proof. The fact that the finitely generated ideals of ℓ^∞ are projective follows from [22, Corollary 2.4]. We will prove that they are principal. Given $\alpha \in \ell^\infty$, set $\nu_\alpha \in \ell^\infty$ be as in (4.4). Notice that ν_α is the partial isometry in the polar decomposition of α . In fact, we have

$$\alpha = \nu_\alpha |\alpha|, \quad |\alpha| = \overline{\nu}_\alpha \alpha.$$

It follows that, for any ideal I in ℓ^∞ , $\alpha \in I$ if and only if $|\alpha| \in I$. Now let I be an ideal of ℓ^∞ generated by $\{\alpha_0, \alpha_1\}$, and set

$$\mu(n) = \max\{|\alpha_0(n)|, |\alpha_1(n)|\}.$$

For $i = 0, 1$, let

$$\gamma_i(n) = \begin{cases} 1/2 & \text{if } |\alpha_0(n)| = |\alpha_1(n)| \\ 1 & \text{if } |\alpha_i(n)| > |\alpha_{1-i}(n)| \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mu = \gamma_0 |\alpha_0| + \gamma_1 |\alpha_1|$; thus $\mu \in I$. Now set

$$\tau_i(n) = \begin{cases} 0 & \text{if } \mu(n) = 0 \\ \frac{\alpha_i(n)}{\mu(n)} & \text{otherwise.} \end{cases}$$

Then $\alpha_i = \tau_i \mu$, ($i = 0, 1$). Notice that $\tau_i \in \ell^\infty$, since $|\tau_i(n)| \leq 1$ for all $n \in \mathbb{N}$, $i = 0, 1$. Therefore, μ generates I . The general case can now be proven by induction on the number of generators. \square

Corollary 6.2.2. *Every ideal of ℓ^∞ is flat.*

Proposition 6.2.3. *Let \mathfrak{A} be a unital Banach algebra and $S \triangleleft \ell^\infty$ a symmetric ideal. Assume that*

$$\alpha \in S \Rightarrow \sqrt{|\alpha|} \in S.$$

Then $S(\mathfrak{A}) \triangleleft \ell^\infty(\mathfrak{A})$ is flat both as a right and as a left $\ell^\infty(\mathfrak{A})$ -module.

Proof. Consider the following homomorphism of $\ell^\infty(\mathfrak{A})$ -modules

$$\mu : \ell^\infty(\mathfrak{A}) \otimes S \rightarrow S(\mathfrak{A}), \quad \mu(\alpha \otimes \beta)_n = \alpha_n \beta_n.$$

We claim that μ is an isomorphism. To prove it is surjective, for $\alpha \in S(\mathfrak{A})$ let ν_α be as in (4.4). Then $\nu_\alpha \in \ell^\infty(\mathfrak{A})$ and

$$\alpha = \mu(\nu_\alpha \otimes \|\alpha\|).$$

Thus μ is surjective. To prove it is also injective, let

$$\eta = \sum_{i=1}^n \alpha^i \otimes \beta^i \in \ker \mu.$$

By Proposition 6.2.1, the ideal $\langle \beta^1, \dots, \beta^n \rangle \triangleleft \ell^\infty$ is principal. Let β be a generator; we may and do choose it so that $\beta = |\beta|$. By bilinearity, we may rewrite η as a single elementary tensor and we have

$$\eta = \alpha \otimes \beta, \quad \alpha\beta = 0.$$

But $\alpha\beta = 0$ implies $\alpha\sqrt{\beta} = 0$, whence

$$\eta = \alpha\sqrt{\beta} \otimes \sqrt{\beta} = 0.$$

Thus the claim is proved. It follows that $S(\mathfrak{A})$ is flat as a left $\ell^\infty(\mathfrak{A})$ -module, since it is the scalar extension of S , which is a flat ℓ^∞ -module by Corollary 6.2.2. The proof that $S(\mathfrak{A})$ is flat on the right is similar. \square

Examples 6.2.4. The hypothesis of Proposition 6.2.3 are satisfied for example when S is either of ℓ^∞^- , c_0 .

Proposition 6.2.5. *Every two-sided ideal of Γ^∞ is flat both as a left and as a right Γ^∞ -module.*

Proof. Let $I \triangleleft \Gamma^\infty$. By Theorem 4.5 there is a symmetric ideal S such that $I = I_S$. Observe that

$$I_S = S \otimes_{\mathcal{P}} \Gamma = S \otimes_{\ell^\infty} \ell^\infty \otimes_{\mathcal{P}} \Gamma = S \otimes_{\ell^\infty} \Gamma^\infty.$$

Thus $I_S \otimes_{\Gamma^\infty} = S \otimes_{\ell^\infty}$ is exact by Corollary 6.2.2. Hence I is flat as a right module and therefore also as a left module, since Γ^∞ is a $*$ -algebra. \square

Remark 6.2.6. We saw in Proposition 4.6 that if k is a field, then $M_\infty k$ is the only proper two-sided ideal of $\Gamma(k)$. Observe that $M_\infty k$ is projective both as a left and as a right module, since it is isomorphic to an infinite sum of copies of the principal ideal generated by the idempotent $E_{1,1}$.

Proposition 6.2.7. *Let \mathfrak{A} be a unital Banach algebra and $S \triangleleft \ell^\infty$ a symmetric ideal as in Proposition 6.2.3. Then $I_{S(\mathfrak{A})}$ is flat both as a left and as a right $\Gamma^\infty(\mathfrak{A})$ -module.*

Proof. By Proposition 5.11 and the proof of Proposition 6.2.3 we have the following canonical isomorphisms of right $\Gamma^\infty(\mathfrak{A})$ -modules

$$I_{S(\mathfrak{A})} = S(\mathfrak{A}) \otimes_{\mathcal{P}} \Gamma = S \otimes_{\ell^\infty} \ell^\infty(\mathfrak{A}) \otimes_{\mathcal{P}} \Gamma = S \otimes_{\ell^\infty} \Gamma^\infty(\mathfrak{A}).$$

This, together with Corollary 6.2.2, prove that $I_{S(\mathfrak{A})}$ is flat as a right $\Gamma^\infty(\mathfrak{A})$ -module. The proof that it is also flat on the left is similar. \square

6.3. Flatness properties of \mathcal{P} . Recall that a \mathbb{Q} -algebra A which is projective as an $A \otimes A^{op}$ -module is called *separable*.

Proposition 6.3.1. *The \mathbb{Q} -algebra \mathcal{P} is a filtering union of separable algebras.*

Proof. We shall show that \mathcal{P} is a union of finite products of copies of \mathbb{Q} , indexed by the finite partitions of \mathbb{N} . Here a finite partition of \mathbb{N} is a finite set $\pi = \{A_1, \dots, A_n\}$ of subsets of \mathbb{N} such that $\mathbb{N} = A_1 \sqcup \dots \sqcup A_n$. We say that a partition $\rho = \{B_1, \dots, B_m\}$ is *finer* than π if the following condition is satisfied:

$$(\forall 1 \leq i \leq m)(\exists j) \quad B_i \subset A_j.$$

Note that if π and π' are any two finite partitions, then

$$\pi \wedge \pi' = \{B \subset \mathbb{N} : (\exists A \in \pi, A' \in \pi') B = A \cap A'\}.$$

is a finite partition and is finer than each of them. Thus the set

$$\text{Part}(\mathbb{N}) = \{\pi \text{ finite partition of } \mathbb{N}\}.$$

is a filtered partially ordered set. If $\pi \in \text{Part}(\mathbb{N})$ has n elements, put

$$\mathcal{P} \supset R_\pi = \bigoplus_{i=1}^n \mathbb{Q}P_{A_i}.$$

Observe that $R_\pi \cong \mathbb{Q}^n$ and that $\mathcal{P} = \bigcup_{\pi} R_\pi$. Hence the proof is completed. \square

Corollary 6.3.2. *\mathcal{P} is a von Neumann regular ring. In other words, every \mathcal{P} -module is flat.*

7. HOMOTOPY INVARIANCE

7.1. Crossed products by the Cohn algebra. The following two elements of Emb will play a central role in what follows

$$\begin{aligned} s_i : \mathbb{N} &\rightarrow \mathbb{N} \quad (i = 1, 2) \\ s_i(m) &= 2m + i - 1. \end{aligned}$$

We have the following relations

$$s_i^\dagger s_j = \delta_{i,j} \quad i = 1, 2. \tag{7.1.1}$$

Following standard conventions, if ν is a word of length l on $\{1, 2\}$, we write $s_\nu = s_{\nu_1} \cdots s_{\nu_l}$ and $s_\nu^\dagger = (s_\nu)^\dagger$. Thus for the empty word we have $s_\emptyset = s_\emptyset^\dagger = 1$. Observe that if μ is of length l then

$$s_\mu(n) = 2^l n + \sum_{i=1}^l (\mu_i - 1) 2^{i-1}. \quad (7.1.2)$$

Put

$$W_2^l = \{ \text{words of length } l \text{ on } \{1, 2\} \}, \quad W_2 = \bigcup_{l=0}^{\infty} W_2^l.$$

We write

$$\mathcal{M}_2 = \{s_\mu(s_\nu)^\dagger : \mu, \nu \in W_2\}.$$

Thus $\mathcal{M}_2 \subset \text{Emb}$ is the inverse submonoid generated by the s_i . Its idempotent submonoid is

$$E(\mathcal{M}_2) = \{s_\nu(s_\nu)^\dagger : \nu \in W_2\}.$$

One checks, using (7.1.2) that $s_\mu s_\nu^\dagger = s_{\mu'} s_{\nu'}^\dagger$ if and only if $\mu = \mu'$ and $\nu = \nu'$. It follows that \mathcal{M}_2 is the universal inverse monoid on generators s_1, s_2 subject to the relations (7.1.1). Write

$$C_2 = \mathbb{Q}[\mathcal{M}_2] \supset \mathcal{P}_2 = \mathbb{Q}[E(\mathcal{M}_2)].$$

The algebra C_2 is the *Cohn algebra* on two generators ([4]). The assignment

$$E_{s_\mu(1), s_\nu(1)} \mapsto s_\mu \left(1 - \sum_{i=1}^2 s_i s_i^\dagger \right) s_\nu^*.$$

defines an isomorphism between M_∞ and the ideal of C_2 generated by $1 - \sum_{i=1}^2 s_i s_i^\dagger$. We shall identify each element of M_∞ with its image in C_2 . If \mathfrak{A} is a bornological algebra and $S \triangleleft \ell^\infty$ is a symmetric ideal, then we can consider the action of \mathcal{M}_2 on $S(\mathfrak{A})$ coming from restriction of the Emb action, and form the crossed product $S(\mathfrak{A}) \# \mathcal{M}_2$. Recall from Section §5 that $S(\mathfrak{A}) \# \mathcal{M}_2 = S(\mathfrak{A}) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathcal{M}_2]$ equipped with the product (5.8). Put

$$S(\mathfrak{A}) \#_{\mathcal{P}_2} C_2 = S(\mathfrak{A}) \# \mathcal{M}_2 / \langle \alpha p \# f - \alpha \# p f : p \in E(\mathcal{M}_2), f \in \mathcal{M}_2 \rangle.$$

As a vector space, $S(\mathfrak{A}) \#_{\mathcal{P}_2} C_2 = S(\mathfrak{A}) \otimes_{\mathcal{P}_2} C_2$; the product is defined as in (5.8). We have an algebra homomorphism

$$\rho : S(\mathfrak{A}) \#_{\mathcal{P}_2} C_2 \rightarrow I_{S(\mathfrak{A})}, \quad \rho(\alpha \# f) = \text{diag}(\alpha) U_f. \quad (7.1.3)$$

Lemma 7.1.4. *The map (7.1.3) is injective.*

Proof. Any nonzero element $x \in C_2$ can be written as a finite sum of nonzero terms

$$x = \sum_{\mu, \nu} \alpha_{\mu, \nu} \# s_\mu s_\nu^\dagger. \quad (7.1.5)$$

Let l be the maximum length of all the multi-indices μ appearing in the expression above. Remark that we may rewrite (7.1.5) as another finite sum

$$x = \sum_{i,j} x_{i,j} \# E_{i,j} + \sum_{l(\mu)=l} \beta_{\mu,\nu} \# s_{\mu} s_{\nu}^{\dagger}. \quad (7.1.6)$$

such that

$$x_{i,j} \neq 0 \Rightarrow i < 2^l. \quad (7.1.7)$$

Indeed this follows from (7.1.2) and from the identities

$$\begin{aligned} s_{\mu} s_{\nu}^{\dagger} &= s_{\mu} (1 - \sum_{i=1}^2 s_i s_i^{\dagger}) s_{\nu}^{\dagger} + \sum_{i=1}^2 s_{\mu i} s_{\nu i}^{\dagger} \\ &= E_{\mu(1),\nu(1)} + \sum_{i=1}^2 s_{\mu i} s_{\nu i}^{\dagger}. \end{aligned}$$

Suppose that the element (7.1.6) is in $\ker \rho$. Observe that $\rho(\chi_{\{i\}} \otimes E_{i,j}) = E_{i,j}$. Hence, we have

$$0 = \sum_{i,j} x_{i,j} E_{i,j} + \sum_{l(\mu)=l,\nu} \text{diag}(\beta_{\mu,\nu}) U_{s_{\mu}} U_{s_{\nu}}^*. \quad (7.1.8)$$

But by (7.1.2), for μ as in (7.1.8), we have

$$\text{ran}(U_{s_{\mu}} U_{s_{\nu}}^*) = \text{span}\{e_n : n = 2^l m + \sum_{i=1}^l (\mu_i - 1) 2^{i-1} \quad m \in \mathbb{N}\}.$$

This together with (7.1.7) imply that each of the summands of (7.1.8) vanishes. Thus

$$x_{i,j} = 0 \text{ and } \text{diag}(\beta_{\mu,\nu}) U_{s_{\mu}} U_{s_{\nu}}^* = 0$$

for all i, j and all μ and ν in (7.1.7). Hence,

$$\emptyset = \text{supp} \beta_{\mu,\nu} \cap (2^l \mathbb{N} + \sum_{i=1}^l (\mu_i - 1) 2^{i-1}) = \text{supp}(s_{\mu} s_{\mu}^{\dagger})_*(\beta_{\mu,\nu}).$$

It follows that $\beta_{\mu,\nu} \# s_{\mu} s_{\nu}^{\dagger} = 0$ and therefore the element (7.1.6) must be zero, finishing the proof. \square

Remark 7.1.9. Let $S \triangleleft \ell^{\infty}$ be a nonzero symmetric ideal and let c_f be as in Example 3.17. Then S contains c_f and thus if we identify $S \#_{\mathcal{P}_2} C_2$ with its image in I_S , we have

$$I_S \supset S \#_{\mathcal{P}_2} C_2 \supset c_f \#_{\mathcal{P}_2} C_2 = M_{\infty}.$$

In particular the completion of $c_0 \#_{\mathcal{P}_2} C_2$ with respect to the operator norm in $\mathcal{B}(\ell^2)$ coincides with the completion of M_{∞} and of I_{c_0} ; it is the ideal $\mathcal{K} = J_{c_0}$ of compact operators. Similarly, for $p \geq 1$ the completion of $\ell^p \#_{\mathcal{P}_2} C_2$ for the p -Schatten norm $\|T\|_p = \text{Tr}(|T|^p)$ coincides with that of I_{ℓ^p} ; it is the Schatten ideal \mathcal{L}^p .

7.2. The Cohn algebra and homotopy invariance. Let \mathbb{V} be a bornological vector space, T a compact Hausdorff topological space, X a metric space, and $1 \geq \lambda > 0$. Put

$$C(T, \mathbb{V}) = \{f : T \rightarrow \mathbb{V} \text{ continuous}\},$$

$$H^\lambda(X, \mathbb{V}) = \{f : X \rightarrow \mathbb{V} \mid \lambda\text{-H\"older continuous}\}.$$

We refer the reader to [13, §2.1.1 and §3.1.4] for the definitions of continuity and H\"older continuity in the bornological setting, as well as for those of the canonical uniform bornologies that the above algebras carry.

Let $S \triangleleft \ell^\infty$ be a symmetric ideal and \mathfrak{A} a bornological algebra. We have a natural inclusion

$$\text{inc} : \mathfrak{A} \subset S(\mathfrak{A}), a \mapsto (a, 0, 0, \dots).$$

Lemma 7.2.1. (cf. [13, Lemma 3.26]) *Let $F : \mathbb{C} - \text{Alg} \rightarrow \mathfrak{A}\mathfrak{b}$ be a split-exact, M_2 -stable functor, \mathfrak{B} a bornological algebra, $\text{ev}_t : C([0, 1], \mathfrak{B}) \rightarrow \mathfrak{B}$ the evaluation map, and $0 < \lambda \leq 1$.*

i)

$$F\left(C([0, 1], \mathfrak{B}) \xrightarrow{\text{ev}_t} \mathfrak{B} \xrightarrow{\text{inc}} c_0(\mathfrak{B}) \xrightarrow{-\#1} c_0(\mathfrak{B}) \#_{\mathcal{P}_2} C_2\right)$$

is independent of t .

ii) *If $p > 1/\lambda$, then*

$$F\left(H^\lambda([0, 1], \mathfrak{B}) \xrightarrow{\text{ev}_t} \mathfrak{A} \xrightarrow{\text{inc}} \ell^p(\mathfrak{B}) \xrightarrow{-\#1} \ell^p(\mathfrak{B}) \#_{\mathcal{P}_2} C_2\right)$$

is independent of t .

Proof. Let S be either c_0 or ℓ^p . In the first case, put $\mathfrak{B}[0, 1] = C([0, 1], \mathfrak{B})$; in the second, let $\lambda > 1/p$ and set $\mathfrak{B}[0, 1] = H^\lambda([0, 1], \mathfrak{B})$. Let

$$\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \supset X = \{(l, k) : k \leq 2^l - 1\}.$$

Let ϕ_+, ϕ_-, ϕ_0^2 and ϕ_-^2 be the homomorphisms $\mathfrak{B}[0, 1] \rightarrow \ell^\infty(X, \mathfrak{B})$ defined in the proof of [13, Lemma 3.26]. One checks that (ϕ_+, ϕ_-) and (ϕ_0^2, ϕ_-^2) are quasi-homomorphisms $\mathfrak{B}[0, 1] \rightarrow S(X, \mathfrak{B})$. Furthermore, it is shown in loc. cit. that there are elements $V, \bar{V} \in \text{Emb}(X)$ such that for

$$\text{inc}_{0,0} : \mathfrak{B} \rightarrow S(X, \mathfrak{B}), \quad \text{inc}_{0,0}(a)_{l,k} = a\delta_{l,0}\delta_{k,0}$$

we have

$$\begin{aligned} F(\text{inc}_{0,0} \circ \text{ev}_0) - F(\text{inc}_{0,0} \circ \text{ev}_1) &= (F(\bar{V}_*) - 1)F(\phi_-, \phi_+) \\ &\quad + (F(V_*) - 1)F(\phi_0^2, \phi_-^2). \end{aligned} \quad (7.2.2)$$

Consider the bijection $\psi : X \rightarrow \mathbb{N}$

$$\psi(l, k) = 2^l + k. \quad (7.2.3)$$

Let s_1, s_2 be the generators (7.1) of C_2 . Let $v, \bar{v} \in \text{Emb}$ be the conjugates of V and \bar{V} under ψ . One checks that, for ρ as in (7.1.3), we have

$$\bar{v} = s_2 \text{ and} \quad (7.2.4)$$

$$U_v = \rho(1 - s_1 s_1^\dagger - s_2 s_2^\dagger + s_2 s_1^\dagger + s_1 s_2^\dagger). \quad (7.2.5)$$

Now recall that $C_2 = \mathbb{Q}[\mathcal{M}_2]$ and write $*$: $C_2 \rightarrow C_2$ for the involution induced by \dagger . It follows from (7.2.5) that the element

$$C_2 \ni f = 1 - s_1 s_1^\dagger - s_2 s_2^\dagger + s_2 s_1^\dagger + s_1 s_2^\dagger \quad (7.2.6)$$

satisfies $f^* f = 1$. Hence if g is any of $1 \# s_2, 1 \# f \in \ell^\infty(\tilde{\mathfrak{B}}) \# C_2$, we have an algebra homomorphism

$$\text{conj}(g) : S(\mathfrak{B}) \# C_2 \rightarrow S(\mathfrak{B}) \# C_2, \quad x \mapsto g x g^*.$$

Moreover, because F is M_2 -stable by assumption and $S(\mathfrak{B}) \# C_2$ is an ideal in $\ell^\infty(\tilde{\mathfrak{B}}) \# C_2$, $F(\text{conj}(g))$ is the identity ([6, Proposition 2.2.6]). Let $\phi_0'^2$, ϕ_-' , ϕ_+' and ϕ_-' be the maps $\mathfrak{B}[0, 1] \rightarrow S(\mathfrak{B})$ obtained from ϕ_0^2 , ϕ_-^2 , ϕ_+ , and ϕ_- after conjugating with U_ψ . Then (7.2.2) gives the identity

$$\begin{aligned} F((\text{incev}_0) \# 1) - F((\text{incev}_1) \# 1) = \\ (F(\text{conj}(1 \# s_2)) - 1)F(\phi_-', \phi_+') + (F(\text{conj}(1 \# f)) - 1)F(\phi_0'^2, \phi_-'^2) = 0. \end{aligned}$$

We have proved that $F((\text{inc} \circ \text{ev}_0) \# 1) = F((\text{inc} \circ \text{ev}_1) \# 1)$. The proposition now follows from the fact that if $t \in [0, 1]$ then ev_t and ev_0 are linearly homotopic. \square

Remark 7.2.7. The key property of C_2 used in the proof of Lemma 7.2.1 is that it contains the elements (7.2.4) and (7.2.6). In fact it is not hard to check that they generate the algebra C_2 . Hence taking crossed product with C_2 may be regarded as the smallest construction which makes the proof given above work.

Remark 7.2.8. If \mathfrak{A} is a C^* -algebra, then $c_0(\mathfrak{A}) = c_0 \tilde{\otimes} \mathfrak{A}$ is the spatial C^* -algebra tensor product. The inclusion $c_0 \subset I_{c_0} \subset \mathcal{K}$ is equivariant for the action of Emb , and so we get a map $c_0(\mathfrak{A}) \#_{\mathcal{P}_2} C_2 \rightarrow \mathfrak{A} \tilde{\otimes} \mathcal{K}$. Composing the latter with the inclusion $\mathfrak{A} \rightarrow c_0(\mathfrak{A}) \#_{\mathcal{P}_2} C_2$ of Lemma 7.2.1 we obtain the map $\iota : \mathfrak{A} \rightarrow \mathfrak{A} \tilde{\otimes} \mathcal{K}$, $a \mapsto a \tilde{\otimes} E_{1,1}$. Hence, the lemma implies that if $F : \mathbb{C} - \text{Alg} \rightarrow \mathfrak{A} \mathfrak{b}$ is split-exact and M_2 -stable, then, for every C^* -algebra \mathfrak{B} , the map

$$F \left(C([0, 1], \mathfrak{B}) \xrightarrow{\text{ev}_t} \mathfrak{B} \xrightarrow{\iota} \mathfrak{B} \tilde{\otimes} \mathcal{K} \right)$$

is independent of t . One can use this to prove that F is homotopy invariant on stable C^* -algebras, thus obtaining a weak version of Higson's homotopy invariance theorem [18, Theorem 3.2.2]. Indeed it suffices to show that $F(\iota)$ is injective if $\mathfrak{B} = \mathfrak{A} \tilde{\otimes} \mathcal{K}$, and this follows from the fact that there is a map

$\mathcal{K} \overset{\sim}{\otimes} \mathcal{K} \rightarrow M_2\mathcal{K}$ (in fact an isomorphism) such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{K} \overset{\sim}{\otimes} \mathcal{K} & \longrightarrow & M_2\mathcal{K} \\ \uparrow \iota & \nearrow E_{1,1} & \\ \mathcal{K} & & \end{array} \quad (7.2.9)$$

Next suppose that \mathfrak{B} is any bornological algebra. Write $\hat{\otimes}$ for the projective tensor product. For each $p \geq 1$ we have a map $\ell^p \hat{\otimes} \mathfrak{B} \rightarrow \ell^p(\mathfrak{B})$. This map is an isomorphism if $p = 1$; using this isomorphism as above, we obtain a map

$$\ell^1(\mathfrak{A}) \#_{\mathcal{P}_2} C_2 \rightarrow \mathfrak{A} \hat{\otimes} \mathcal{L}^1.$$

In general $\ell^p \hat{\otimes} \mathfrak{A} \rightarrow \ell^p(\mathfrak{A})$ is not an isomorphism. Note, however, that for every $p \geq 1$, the quotient $\ell^p(\mathfrak{A})/\ell^1(\mathfrak{A})$ is a nilpotent ring. Assume that the functor F is *strongly nilinvariant* in the sense that if $f : A \rightarrow B$ is a homomorphism with nilpotent kernel, and such that $f(A) \triangleleft B$ and $B/f(A)$ is nilpotent, then $F(f)$ is an isomorphism. Then $F(\ell^1(\mathfrak{A}) \#_{\mathcal{P}_2} C_2) \rightarrow F(\ell^p(\mathfrak{A}) \#_{\mathcal{P}_2} C_2)$ and $F(\mathfrak{A} \hat{\otimes} \mathcal{L}^1) \rightarrow F(\mathfrak{A} \hat{\otimes} \mathcal{L}^p)$ are isomorphisms for all $p \geq 1$. Moreover we also have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}^1 \hat{\otimes} \mathcal{L}^1 & \longrightarrow & M_2\mathcal{L}^1 \\ \uparrow \iota & \nearrow E_{1,1} & \\ \mathcal{L}^1 & & \end{array} \quad (7.2.10)$$

Let \mathbf{BAlg} be the category of bornological algebras and bounded homomorphisms. Using Lemma 7.2.1 together with diagram (7.2.10) above and proceeding as before, one shows that if F is split-exact, M_2 -stable, and strongly nilinvariant, then the functor

$$\mathbf{BAlg} \rightarrow \mathfrak{Ab}, \quad \mathfrak{A} \mapsto F(\mathfrak{A} \hat{\otimes} \mathcal{L}^1),$$

is invariant under Hölder-continuous homotopies. This gives a (weak) bornological version of [10, Theorem 6.1.6]. Observe that the stability properties (7.2.9) and (7.2.10) play a crucial role in the arguments above. We do not have an analogue stability result for the uncompleted algebras $c_0(\mathfrak{A}) \#_{\mathcal{P}_2} C_2$ and $\ell^1(\mathfrak{A}) \#_{\mathcal{P}_2} C_2$. In the next subsection we shall prove a version of stability for crossed products with Γ . This will enable us to prove a homotopy invariance theorem in the following subsection.

7.3. Stability.

Lemma 7.3.1.

- i) *There is a natural isomorphism $\Gamma(\mathbb{N} \sqcup \mathbb{N}) \cong M_2\Gamma$.*
- ii) *Let \mathfrak{A} be a bornological algebra and $S \triangleleft \ell^\infty$ a symmetric ideal. Then $I_{S(\mathbb{N} \sqcup \mathbb{N}, \mathfrak{A})} \cong M_2 I_{S(\mathfrak{A})}$.*

Proof. Let $p_1, p_2 \in \text{Emb}(\mathbb{N} \sqcup \mathbb{N})$ be the inclusions of each of the copies of \mathbb{N} . If $f \in \text{Emb}(\mathbb{N} \sqcup \mathbb{N})$, then $p_i f p_j$ identifies in the obvious way with an element $f_{i,j} \in \text{Emb}$. One checks that the map

$$\text{Emb}(\mathbb{N} \sqcup \mathbb{N}) \rightarrow M_2\Gamma, \quad f \mapsto (U_{f_{ij}})$$

is multiplicative. Hence it induces a homomorphism

$$\mathbb{Q}[\text{Emb}(\mathbb{N} \sqcup \mathbb{N})] \rightarrow M_2\Gamma.$$

One checks further that this map kills the ideal (5.3), and thus descends to a homomorphism

$$\phi : \Gamma(\mathbb{N} \sqcup \mathbb{N}) \rightarrow M_2\Gamma, \quad \phi(a)_{ij} = U_{p_i} a U_{p_j}. \quad (7.3.2)$$

Observe that $E_{i,j} U_f$ is in the image of (7.3.2) for all $f \in \text{Emb}$. It follows that (7.3.2) is surjective. Moreover because U_{p_1}, U_{p_2} are orthogonal idempotents with $U_{p_1} + U_{p_2} = 1$, $a \in \Gamma(\mathbb{N} \sqcup \mathbb{N})$ is zero if and only if $U_{p_i} a U_{p_j} = 0$ for $1 \leq i, j \leq 2$. Hence (7.3.2) is an isomorphism; this proves part i). To prove part ii) one begins by observing that the assignment $\alpha \mapsto (\alpha p_1, \alpha p_2)$ defines isomorphisms $S(\mathbb{N} \sqcup \mathbb{N}) \xrightarrow{\cong} S(\mathbb{N}) \oplus S(\mathbb{N})$ and $\mathcal{P}(\mathbb{N} \sqcup \mathbb{N}) \xrightarrow{\cong} \mathcal{P}(\mathbb{N}) \oplus \mathcal{P}(\mathbb{N})$. Next, note that if we regard $M_2\Gamma$ as a $\mathcal{P} \oplus \mathcal{P}$ -module via the diagonal inclusion, we have an isomorphism of abelian groups

$$\begin{aligned} (S(\mathfrak{A}) \oplus S(\mathfrak{A})) \otimes_{\mathcal{P} \oplus \mathcal{P}} M_2(\Gamma) &\cong M_2(S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma) \\ (\alpha_1, \alpha_2) \otimes x &\mapsto \sum_{1 \leq i, j \leq 2} \alpha_i \# x_{i,j} \otimes E_{i,j}. \end{aligned}$$

Finally one checks that the algebra homomorphism

$$\begin{aligned} S(\mathbb{N} \sqcup \mathbb{N}, \mathfrak{A}) \#_{\mathcal{P}(\mathbb{N} \sqcup \mathbb{N})} \Gamma(\mathbb{N} \sqcup \mathbb{N}) &\rightarrow M_2(S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma) \\ \alpha \# x &\mapsto \sum_{1 \leq i, j \leq 2} \alpha p_i \# U_{p_i} x U_{p_j} \otimes E_{i,j} \end{aligned}$$

coincides with the following composite of isomorphisms of abelian groups

$$\begin{aligned} S(\mathbb{N} \sqcup \mathbb{N}, \mathfrak{A}) \#_{\mathcal{P}(\mathbb{N} \sqcup \mathbb{N})} \Gamma(\mathbb{N} \sqcup \mathbb{N}) &\cong (S(\mathfrak{A}) \oplus S(\mathfrak{A})) \otimes_{\mathcal{P} \oplus \mathcal{P}} M_2(\Gamma) \\ &\cong M_2(S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma). \end{aligned}$$

□

Let \mathfrak{A} be a bornological algebra and let $\iota : \ell^\infty(\mathfrak{A}) \rightarrow \ell^\infty(\mathbb{N} \times \mathbb{N}, \mathfrak{A})$ be the inclusion

$$\iota(\alpha)(m, n) = \alpha_m \delta_{1,n}.$$

Also let $S \triangleleft \ell^\infty$ be a symmetric ideal; put

$$\begin{aligned} j : S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma &\rightarrow S(\mathbb{N} \times \mathbb{N}, \mathfrak{A}) \#_{\mathcal{P}(\mathbb{N} \times \mathbb{N})} \Gamma(\mathbb{N} \times \mathbb{N}) \\ j(\alpha \# U_f) &= \iota(\alpha) \# (U_f \times \chi_{\{1\}}). \end{aligned} \quad (7.3.3)$$

Proposition 7.3.4. *Let \mathfrak{A} be a bornological algebra and $S \triangleleft \ell^\infty$ a symmetric ideal. Then any M_2 -stable functor $F : \mathbb{C} - \text{Alg} \rightarrow \mathfrak{Ab}$ sends the map j of (7.3.3) to a split monomorphism.*

Proof. Choose a bijection $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \sqcup \mathbb{N}$ sending $\mathbb{N} \times \{1\}$ bijectively onto the first copy of \mathbb{N} . This bijection induces an isomorphism

$$S(\mathbb{N} \times \mathbb{N}, \mathfrak{A}) \#_{\mathcal{P}(\mathbb{N} \times \mathbb{N})} \Gamma(\mathbb{N} \times \mathbb{N}) \xrightarrow{\cong} S(\mathbb{N} \sqcup \mathbb{N}, \mathfrak{A}) \#_{\mathcal{P}(\mathbb{N} \sqcup \mathbb{N})} \Gamma(\mathbb{N} \sqcup \mathbb{N}).$$

Composing this map with the isomorphism of Lemma 7.3.1, we obtain an isomorphism which fits into a commutative diagram

$$\begin{array}{ccc} S(\mathbb{N} \times \mathbb{N}, \mathfrak{A}) \#_{\mathcal{P}(\mathbb{N} \times \mathbb{N})} \Gamma(\mathbb{N} \times \mathbb{N}) & \xrightarrow{\sim} & M_2(S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma) \\ \uparrow j & \nearrow E_{1,1} \otimes - & \\ S(\mathfrak{A}) \#_{\mathcal{P}} \Gamma & & \end{array}$$

This concludes the proof. \square

7.4. A homotopy invariance theorem. Let $f_0, f_1 : \mathfrak{A} \rightarrow \mathfrak{B}$ be homomorphisms of bornological algebras and $0 < \lambda \leq 1$. A λ -Hölder continuous homotopy from f_0 to f_1 is a homomorphism $H : \mathfrak{A} \rightarrow H^\lambda([0, 1], \mathfrak{B})$ such that $\text{ev}_i H = f_i$ ($i = 0, 1$). We say that a functor $F : \text{BAlg} \rightarrow \mathfrak{Ab}$ is *invariant under λ -Hölder homotopies* if it maps λ -Hölder homotopic homomorphisms to equal maps.

Theorem 7.4.1. *Let $F : \mathbb{C} - \text{Alg} \rightarrow \mathfrak{Ab}$ be a split-exact, M_2 -stable functor.*

i) *The functor*

$$\text{BAlg} \rightarrow \mathfrak{Ab}, \mathfrak{B} \mapsto F(I_{c_0(\mathfrak{B})})$$

is invariant under continuous homotopies.

ii) *If $1 \geq \lambda > 0$ and $p > 1/\lambda$, then the functor*

$$\text{BAlg} \rightarrow \mathfrak{Ab}, \mathfrak{B} \mapsto F(I_{\ell^p(\mathfrak{B})})$$

is invariant under λ -Hölder homotopies.

Proof. Let \mathfrak{A} be a bornological algebra. We adopt the notations of the proof of Lemma 7.2.1. Thus S will be either of c_0 or ℓ^p , and $\mathfrak{A}[0, 1]$ will stand for $C([0, 1], \mathfrak{A})$ in the first case, and for $H^\lambda([0, 1], \mathfrak{A})$ in the second. By the argument of the proof of Lemma 7.2.1 applied to the functor

$$G = F(S(-) \#_{\mathcal{P}} \Gamma), \quad (7.4.2)$$

we have the following identity

$$\begin{aligned} G(\text{inc})(G(\text{ev}_0)) - G(\text{ev}_1) &= (G((s_2)_*) - 1)G(\phi'_-, \phi'_+) \\ &\quad + (G(f_*) - 1)G(\phi_0'^2, \phi_-'^2). \end{aligned} \quad (7.4.3)$$

Now if $h \in \text{Emb}$ then $G(h_*)$ is the result of applying F to the map

$$S(h_*) \#_{\mathcal{P}} \Gamma : S(S(\mathfrak{A})) \#_{\mathcal{P}} \Gamma \rightarrow S(S(\mathfrak{A})) \#_{\mathcal{P}} \Gamma.$$

Here the crossed product is taken with respect to the action on the external S . In addition, we consider the action of Γ on the inner S and take the crossed product again; we write $(S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma)\#_{\mathcal{P}}\Gamma$ for the resulting algebra. We have an inclusion

$$\text{inc}' = -\#1 : S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma \subset (S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma)\#_{\mathcal{P}}\Gamma$$

and a commutative diagram

$$\begin{array}{ccc} S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma & \xrightarrow{S(h_*)\#_{\mathcal{P}}\Gamma} & S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma \\ \text{inc}' \downarrow & & \downarrow \text{inc}' \\ (S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma)\#_{\mathcal{P}}\Gamma & \xrightarrow{\text{conj}(1\#U_h)} & (S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma)\#_{\mathcal{P}}\Gamma \end{array}$$

Because F is M_2 -stable, $F(\text{conj}(1\#U_h))$ is the identity map, since

$$S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma\#_{\mathcal{P}}\Gamma \triangleleft (\ell^\infty(\ell^\infty(\mathfrak{A}))\#_{\mathcal{P}}\Gamma)\#_{\mathcal{P}}\Gamma \ni 1\#U_h.$$

Hence, by (7.4.3),

$$\begin{aligned} F(\text{inc}'(S(\text{inc})\#_{\mathcal{P}}\Gamma))(F(S(\text{ev}_0)\#_{\mathcal{P}}\Gamma) - F(S(\text{ev}_1)\#_{\mathcal{P}}\Gamma)) = \\ F(\text{inc}')(G((s_2)_*) - 1)G(\phi'_-, \phi'_+) \\ + F(\text{inc}')(G(f_*) - 1)G(\phi'^2_0, \phi'^2_-) = 0. \end{aligned} \quad (7.4.4)$$

We have to show that

$$F(\text{inc}'(S(\text{inc})\#_{\mathcal{P}}\Gamma)) \quad (7.4.5)$$

is injective. Observe that we have a natural isomorphism

$$\mu : S(S(\mathfrak{A})) \xrightarrow{\cong} S(\mathbb{N} \times \mathbb{N}, \mathfrak{A}), \quad \mu(\alpha)_{m,n} = (\alpha_n)_m. \quad (7.4.6)$$

For $h \in \text{Emb}$ the isomorphism (7.4.6) transforms $S(h_*)$ into the action of $1 \times h \in \text{Emb}(\mathbb{N} \times \mathbb{N})$, and h_*S into that of $h \times 1$. Hence we have a map

$$\begin{aligned} \text{inc}'' : (S(S(\mathfrak{A}))\#_{\mathcal{P}}\Gamma)\#_{\mathcal{P}}\Gamma &\rightarrow S(\mathbb{N} \times \mathbb{N})\#_{\mathcal{P}(\mathbb{N} \times \mathbb{N})}\Gamma(\mathbb{N} \times \mathbb{N}) \\ \text{inc}''(\alpha\#U_g\#U_h) &= \mu(\alpha)\#(U_{g \times h}). \end{aligned}$$

Observe that the composite

$$\text{inc}''\text{inc}'(S(\text{inc})\#_{\mathcal{P}}\Gamma) = j$$

is the map of (7.3.3). By Proposition 7.3.4, this implies that the map (7.4.5) is injective, concluding the proof. \square

8. K -THEORY

8.1. Homotopy algebraic K -theory. Let $0 < p \leq \infty$. Put

$$\ell^{p-} = \bigcup_{q < p} \ell^q.$$

For $0 < p < \infty$ we also consider

$$\ell^{p+} = \bigcap_{q>p} \ell^q.$$

We say that a functor $F : \mathbf{BAlg} \rightarrow \mathbf{Ab}$ is *Hölder homotopy invariant* if it is invariant under λ -Hölder homotopies for all $0 < \lambda \leq 1$. Recall from [13, §2] that a bornological algebra is called a *local Banach algebra* if it is a filtering union of Banach subalgebras. Similarly we say that a bornological algebra is a *local C^* -algebra* if it is a filtering union of C^* -subalgebras. If $\mathfrak{A} = \bigcup_{\lambda} \mathfrak{A}_{\lambda}$ and $\mathfrak{B} = \bigcup_{\mu} \mathfrak{B}_{\mu}$ are local C^* -algebras, we define their spatial tensor product as the algebraic colimit of the spatial tensor products $\mathfrak{A}_{\lambda} \tilde{\otimes} \mathfrak{B}_{\mu}$; $\mathfrak{A} \tilde{\otimes} \mathfrak{B} = \operatorname{colim}_{\lambda, \mu} \mathfrak{A}_{\lambda} \tilde{\otimes} \mathfrak{B}_{\mu}$. For the projective tensor product of bornological spaces (and of bornological algebras) see [13, §2.1.2]. In the next theorem and elsewhere we write KV for Karoubi-Villamayor's K -theory.

Theorem 8.1.1. *Let S be one of ℓ^p , ℓ^{p+} ($0 < p < \infty$) or ℓ^{p-} ($0 < p \leq \infty$).
i) The functor $\mathbf{BAlg} \rightarrow \mathbf{Ab}$, $\mathfrak{A} \mapsto KH_*(I_{\ell^1(\mathfrak{A})})$ is Hölder homotopy invariant and we have $KH_*(I_S(\mathfrak{A})) = KH_*(I_{\ell^1(\mathfrak{A})})$ for all S as above.
ii) For every bornological algebra \mathfrak{A}*

$$KH_n(I_{\ell^1(\mathfrak{A})}) = \begin{cases} KV_n(I_{\ell^1(\mathfrak{A})}) & n \geq 1 \\ K_n(I_{\ell^1(\mathfrak{A})}) & n \leq 0. \end{cases}$$

iii) If \mathfrak{A} is a local Banach algebra and $n \geq 0$, then there is a natural split monomorphism $K_n^{\operatorname{top}}(\mathfrak{A}) \rightarrow KH_n(I_{\ell^1(\mathfrak{A})})$.

Proof. Recall that KH satisfies excision, vanishes on nilpotent rings and commutes with filtering colimits ([30]). On the other hand, $\ell^q(\mathfrak{A})/\ell^p(\mathfrak{A})$ is nilpotent for $p < q < \infty$ and

$$\ell^{r-}(\mathfrak{A}) = \operatorname{colim}_{s < r} \ell^s(\mathfrak{A}) \quad (0 < r \leq \infty).$$

It follows that $KH_*(I_S(\mathfrak{A})) = KH_*(I_{\ell^1(\mathfrak{A})})$ for all S as in the theorem. Recall also that KH is M_2 -stable. Hence $KH_*(I_{\ell^1(-)}) = KH_*(I_{\ell^p(-)})$ is Hölder-homotopy invariant, by Theorem 7.4.1. This proves i). By [30, Proposition 1.5] (see also [6, Proposition 5.2.3]), in order to prove ii) it suffices to show that $I_{\ell^1(\mathfrak{A})}$ is K_0 -regular. By definition, a ring A is K_0 -regular if for each $n \geq 1$ the canonical map

$$K_0(A) \rightarrow K_0(A[t_1, \dots, t_n])$$

is an isomorphism. This is equivalent to the requirement that for $\underline{t} = (t_1, \dots, t_n)$, the map

$$\epsilon : A[\underline{t}] \rightarrow A[\underline{t}], \quad \epsilon(f) = f(0)$$

induce an isomorphism in K_0 . Observe that

$$\begin{aligned} I_{\ell^1(\mathfrak{A})}[\underline{t}] &= (\ell^1(\mathfrak{A}) \#_{\mathcal{P}\Gamma})[\underline{t}] \\ &= (\ell^1(\mathfrak{A})[\underline{t}]) \#_{\mathcal{P}\Gamma}. \end{aligned} \tag{8.1.2}$$

Also note that, for the projective tensor product,

$$\begin{aligned}\ell^1(C^\infty([0, 1], \mathfrak{A})) &= \ell^1 \hat{\otimes} C^\infty([0, 1], \mathbb{C}) \hat{\otimes} \mathfrak{A} \\ &= C^\infty([0, 1], \ell^1(\mathfrak{A})).\end{aligned}\tag{8.1.3}$$

Next we borrow an argument from [24, Proposition 3.4]. Consider the homomorphism

$$\begin{aligned}\phi : C^\infty([0, 1], \ell^1(\mathfrak{A}))[\underline{t}] &\rightarrow C^\infty([0, 1], \ell^1(\mathfrak{A}))[\underline{t}] \\ \phi(f)(s, \underline{t}) &= f(s, s\underline{t}).\end{aligned}$$

Using the identifications (8.1.2) and (8.1.3) we have a diagram

$$\begin{array}{ccc} I_{\ell^1(C^\infty([0,1], \mathfrak{A}))}[\underline{t}] & \xrightarrow{\phi \# \Gamma} & I_{\ell^1(C^\infty([0,1], \mathfrak{A}))}[\underline{t}] \\ \text{inc} \uparrow & \searrow \epsilon & \downarrow \begin{matrix} s=0 \\ s=1 \end{matrix} \\ I_{\ell^1(\mathfrak{A})}[\underline{t}] & \xrightarrow{\quad \quad} & I_{\ell^1(\mathfrak{A})}[\underline{t}] \\ & \searrow 1 & \end{array}$$

One checks that both the outer and the inner square commute. By Theorem 7.4.1, $K_0(\text{ev}_{s=0} \# \Gamma) = K_0(\text{ev}_{s=1} \# \Gamma)$. It follows that $K_0(\epsilon)$ is the identity; this proves ii). Next assume that \mathfrak{A} is a local Banach algebra; then $K_0^{\text{top}}(\mathfrak{A}) = K_0(\mathfrak{A})$. On the other hand, by universal property of the crossed product, we have a map

$$I_{\ell^1(\mathfrak{A})} = (\ell^1 \hat{\otimes} \mathfrak{A}) \#_{\mathcal{P}} \Gamma \rightarrow \mathcal{L}^1 \hat{\otimes} \mathfrak{A}.\tag{8.1.4}$$

Composing this map with the inclusion

$$\mathfrak{A} \rightarrow I_{\ell^1(\mathfrak{A})}, \quad a \mapsto aE_{1,1},\tag{8.1.5}$$

we obtain the map

$$\mathfrak{A} \rightarrow \mathcal{L}^1 \hat{\otimes} \mathfrak{A}, \quad a \mapsto a \hat{\otimes} E_{1,1}.\tag{8.1.6}$$

Since the latter map induces an isomorphism in K_0 , it follows that (8.1.5) induces a split monomorphism $K_0(\mathfrak{A}) \rightarrow K_0(I_{\ell^1(\mathfrak{A})})$. Thus we have established iii) for $n = 0$. For the case $n \geq 1$, we consider the simplicial algebras of C^∞ functions on the topological standard simplices and of polynomial functions on the algebraic standard simplices:

$$\Delta^{\text{dif}} : [n] \mapsto C^\infty(\Delta^n)$$

and

$$\Delta^{\text{alg}} : [n] \mapsto \mathbb{C}[t_0, \dots, t_n] / \langle \sum t_i - 1 \rangle.$$

Set

$$\begin{aligned}\Delta^{\text{dif}} \mathfrak{A} &= \Delta^{\text{dif}} \hat{\otimes} \mathfrak{A} \text{ and} \\ \Delta^{\text{alg}} \mathfrak{A} &= \Delta^{\text{alg}} \otimes_{\mathbb{C}} \mathfrak{A}.\end{aligned}$$

For $n \geq 1$, we have

$$\begin{aligned} K_n^{\text{top}}(\mathfrak{A}) &= \pi_n BGL(\Delta^{\text{dif}} \mathfrak{A}), \\ KV_n(\mathfrak{A}) &= \pi_n BGL(\Delta^{\text{alg}} \mathfrak{A}). \end{aligned}$$

Hence for $KV(\mathfrak{A}) = BGL(\Delta^{\text{alg}} \mathfrak{A})$, there is a map

$$K_n^{\text{top}}(\mathfrak{A}) \rightarrow \pi_n(KV(\Delta^{\text{dif}}(\mathfrak{A}))).$$

Composing the latter map with that induced by the inclusion (8.1.5), and using parts i) and ii), we get a homomorphism

$$K_n^{\text{top}}(\mathfrak{A}) \rightarrow \pi_n KV(I_{\ell^1(\Delta^{\text{dif}} \mathfrak{A})}) \cong KV_n(I_{\ell^1(\mathfrak{A})}) = KH_n(I_{\ell^1(\mathfrak{A})}). \quad (8.1.7)$$

Composing (8.1.7) with the homomorphism induced by (8.1.4) we obtain

$$K_n^{\text{top}}(\mathfrak{A}) \rightarrow KH_n(\mathcal{L}^1 \hat{\otimes} \mathfrak{A}). \quad (8.1.8)$$

But by [10, Theorem 6.2.1] the comparison map

$$KH_n(\mathcal{L}^1 \hat{\otimes} \mathfrak{A}) \rightarrow K_n^{\text{top}}(\mathcal{L}^1 \hat{\otimes} \mathfrak{A})$$

is an isomorphism. One checks that the latter map composed with (8.1.8) is equivalent to that induced by (8.1.6). But (8.1.6) induces an isomorphism in K^{top} of local Banach algebras. This proves that (8.1.7) is a split monomorphism, concluding the proof. \square

Theorem 8.1.9.

- i) The functor $\text{BAlg} \rightarrow \mathfrak{Ab}$, $\mathfrak{A} \mapsto KH_*(I_{c_0(\mathfrak{A})})$ is invariant under continuous homotopies.
- ii) For every bornological algebra \mathfrak{A}

$$KH_n(I_{c_0(\mathfrak{A})}) = \begin{cases} KV_n(I_{c_0(\mathfrak{A})}) & n \geq 1 \\ K_n(I_{c_0(\mathfrak{A})}) & n \leq 0. \end{cases}$$

- iii) If \mathfrak{A} is a local C^* -algebra and $n \geq 0$, then there is a natural split monomorphism $K_n^{\text{top}}(\mathfrak{A}) \rightarrow KH_n(I_{c_0(\mathfrak{A})})$.

Proof. As in Theorem 8.1.1, part i) follows from Theorem (7.4.1). To prove part ii), first observe that

$$\begin{aligned} c_0(C([0, 1], \mathfrak{A})) &= C_0(\mathbb{N}, C([0, 1], \mathfrak{A})) \\ &= C([0, 1], c_0(\mathfrak{A})). \end{aligned}$$

Then use the argument of the proof of part ii) of Theorem 8.1.1. To prove part iii) first observe that if \mathfrak{A} is a local C^* -algebra, then for the spatial tensor product,

$$c_0(\mathfrak{A}) = c_0 \tilde{\otimes} \mathfrak{A}.$$

Hence if $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ is the C^* -algebra of compact operators then the map $\mathfrak{A} \rightarrow \mathfrak{A} \tilde{\otimes} \mathcal{K}$, $a \rightarrow a \otimes E_{1,1}$ factors through $I_{c_0(\mathfrak{A})}$. Taking this into account, using the fact that, by [26, Theorem 10.9] and [24, Proposition 3.4], the comparison map $KH_*(\mathfrak{A} \tilde{\otimes} \mathcal{K}) \rightarrow K_*^{\text{top}}(\mathfrak{A} \tilde{\otimes} \mathcal{K})$ is an isomorphism, and

substituting continuous functions for C^∞ functions, we may now proceed as in the proof of part iii) of Theorem 8.1.1. \square

Remark 8.1.10. The argument of the proofs of part iii) of Theorems 8.1.1 and 8.1.9 does not work for $n < 0$. Indeed, K_n and K_n^{top} do not agree for such n , not even on algebras on which the former is homotopy invariant. For example negative K -theory is homotopy invariant on commutative C^* -algebras ([11, Theorem 1.2]) yet $K_n(\mathbb{C}) = 0$ for $n < 0$, while $K_{2m}^{\text{top}}(\mathbb{C}) = \mathbb{Z}$ for $m \in \mathbb{Z}$.

Remark 8.1.11. The argument of the proof of Theorem 8.1.1 shows that if \mathfrak{A} is a local Banach algebra then $\mathfrak{A} \rightarrow \mathfrak{A} \hat{\otimes} \mathcal{L}^1$ factors through $I_{\ell^1(\mathfrak{A})}$ and the map

$$KH_n(I_{\ell^1(\mathfrak{A})}) \rightarrow KH_n(\mathfrak{A} \hat{\otimes} \mathcal{L}^1) = K_*^{\text{top}}(\mathfrak{A})$$

is onto for $n \geq 0$. Similarly the argument of the proof of 8.1.9 shows that for \mathfrak{A} a local C^* -algebra maps $\mathfrak{A} \rightarrow \mathfrak{A} \tilde{\otimes} \mathcal{K}$ factors through $I_{c_0(\mathfrak{A})}$ and

$$KH_n(I_{c_0(\mathfrak{A})}) \rightarrow KH_n(\mathfrak{A} \tilde{\otimes} \mathcal{K}) = K_*^{\text{top}}(\mathfrak{A})$$

is onto for $n \geq 0$.

8.2. K -theory and cyclic homology.

Theorem 8.2.1. *Let \mathfrak{A} be a bornological algebra and let S be c_0 , ℓ^p , ℓ^{p+} ($0 < p < \infty$), or ℓ^{p-} ($0 < p \leq \infty$). Then there are long exact sequences ($n \in \mathbb{Z}$)*

$$KH_{n+1}(I_S(\mathfrak{A})) \longrightarrow HC_{n-1}(I_S(\mathfrak{A})) \quad (8.2.2)$$

$$\begin{array}{c} \downarrow \\ KH_n(I_S(\mathfrak{A})) \longleftarrow K_n(I_S(\mathfrak{A})) \end{array}$$

and

$$KH_{n+1}(I_S(\mathfrak{A})) \longrightarrow HC_{n-1}(\Gamma^\infty(\mathfrak{A}) : I_S(\mathfrak{A})) \quad (8.2.3)$$

$$\begin{array}{c} \downarrow \\ KH_n(I_S(\mathfrak{A})) \longleftarrow K_n(\Gamma^\infty(\mathfrak{A}) : I_S(\mathfrak{A})) \end{array}$$

Proof. Let $K^{\text{nil}} = \text{hofi}(K \rightarrow KH)$ be the homotopy fiber of the comparison map. By [6, diagram (86)], there is a natural map $\nu : K^{\text{nil}}(A) \rightarrow HC(A)[-1]$, defined for every \mathbb{Q} -algebra A . Write $K^{\text{ninf}} = \text{hofi}(\nu)$; by [7, Proposition 8.2.4] K^{ninf} is excisive, M_2 -stable and nilinvariant, and K_*^{ninf} commutes with filtering colimits. Hence to prove the theorem it suffices to show that

$$K_*^{\text{ninf}}(I_S(\mathfrak{A})) = 0. \quad (8.2.4)$$

Note also that if $S \neq c_0$, then

$$K_*^{\text{ninf}}(I_S(\mathfrak{A})) = K_*^{\text{ninf}}(I_{\ell^1(\mathfrak{A})})$$

by the same argument as that used in the proof of Theorem 8.1.1 to prove the analogue assertion for KH . Thus we may assume from now on that $S \in \{c_0, \ell^1\}$. By [10, Proposition 3.1.4], to prove (8.2.4) it suffices to show that $I_{S(\mathfrak{A})}$ is K^{inf} -regular. Here K^{inf} is infinitesimal K -theory; by [5] it is excisive and M_2 -stable. Hence, the same argument as that used in the proof of Theorems 8.1.1 and 8.1.9 to prove that $I_{S(\mathfrak{A})}$ is K_0 -regular applies to show that it is also K^{inf} -regular. This completes the proof. \square

Remark 8.2.5. By Examples 6.1.8, we have

$$KH_*(\Gamma^\infty(\mathfrak{A})) = HC_*(\Gamma^\infty(\mathfrak{A})) = K_*(\Gamma^\infty(\mathfrak{A})) = 0$$

for unital \mathfrak{A} . Hence in the unital case, the second sequence of Theorem 8.2.1 can be equivalently expressed in terms of the quotient $\Gamma^\infty(\mathfrak{A})/I_{S(\mathfrak{A})}$; we have a long exact sequence

$$\begin{array}{ccc} KH_{n+1}(\Gamma^\infty(\mathfrak{A})/I_{S(\mathfrak{A})}) & \longrightarrow & HC_{n-1}(\Gamma^\infty(\mathfrak{A})/I_{S(\mathfrak{A})}) \\ & & \downarrow \\ KH_n(\Gamma^\infty(\mathfrak{A})/I_{S(\mathfrak{A})}) & \longleftarrow & K_n(\Gamma^\infty(\mathfrak{A})/I_{S(\mathfrak{A})}) \end{array} \quad (8.2.6)$$

8.3. Excision. A ring A is called *K-excisive* if for every ideal embedding $A \triangleleft B$ the map $K_*(A) \rightarrow K_*(B : A)$ is an isomorphism. It was proved by Suslin and Wodzicki ([26, Theorem C]) that if a ring A satisfies the following property then it is *K-excisive*.

$$\forall a \in A^{\oplus n}, \exists b \in A^{\oplus n}, c, d \in A, \text{ such that } a = cdb \text{ and such that}$$

$$(0 :_A d)_r := \{v \in A : dv = 0\} = (0 :_A cd)_r.$$

The right ideal $(0 :_A d)_r$ is called the *right annihilator* of d in A . The property above is the so-called *left triple factorization property* (TFP). A ring is *K-excisive* if and only if its opposite ring A^{op} is, so rings satisfying the right TFP are excisive also. Further results of Wodzicki ([31, Theorems 1.1 and 3.1]) and of Suslin-Wodzicki ([26, Theorem B]) establish that a \mathbb{Q} -algebra A is *K-excisive* if and only if it is excisive for cyclic homology and that this happens if and only if the *bar complex* $(C_*^{\text{bar}}(A), b')$ is exact. Here

$$b' : C_{n+1}^{\text{bar}}(A) = A^{\otimes n+2} \rightarrow A^{\otimes n+1} = C_n^{\text{bar}}(A) \quad (n \geq 0)$$

$$b'(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

The tensor products above are taken over \mathbb{Z} or, equivalently, over \mathbb{Q} , since A is assumed to be a \mathbb{Q} -algebra. The \mathbb{Q} -algebras whose bar homology vanishes—that is, the *K-excisive* ones—are also called *H-unital*.

Proposition 8.3.1. *Let \mathfrak{A} be a bornological algebra and $S \triangleleft \ell^\infty$ a symmetric ideal. Assume that $S(\mathfrak{A})$ has the (left or right) triple factorization property. Then $I_{S(\mathfrak{A})}$ is *K-excisive*. In particular, the exact sequences (8.2.2) and (8.2.3) are isomorphic in this case.*

Proof. Assume that $S(\mathfrak{A})$ has the left TFP. We have to prove that $I_{S(\mathfrak{A})}$ is H -unital. Let $n \geq 0$ and let $z \in C_n^{bar}(I_{S(\mathfrak{A})})$ be a cycle. We may write

$$z = \sum_{i=1}^m \text{diag}(\alpha^{0,i})U_{f_{0,i}} \otimes \cdots \otimes \text{diag}(\alpha^{n,i})U_{f_{n,i}},$$

where $\text{supp}(\alpha^{j,i}) = \text{ran}(f_{j,i})$ for all i, j . By TFP, there are elements γ, δ and β^1, \dots, β^m in $S(\mathfrak{A})$ such that $\alpha^{0,i} = \gamma\delta\beta^i$ ($1 \leq i \leq m$), and such that

$$(0 :_{S(\mathfrak{A})} \gamma\delta)_r = (0 :_{S(\mathfrak{A})} \delta)_r. \quad (8.3.2)$$

Now observe that if $\theta \in S(\mathfrak{A})$ then, by Proposition 3.14,

$$(0 :_{I_{S(\mathfrak{A})}} \text{diag}(\theta))_r = \{T \in I_{S(\mathfrak{A})} : (\forall j) \ T_{*,j} \in (0 :_{S(\mathfrak{A})} \theta)_r\}.$$

Hence, (8.3.2) implies that

$$(0 :_{I_{S(\mathfrak{A})}} \text{diag}(\gamma\delta))_r = (0 :_{I_{S(\mathfrak{A})}} \text{diag}(\delta))_r. \quad (8.3.3)$$

Put

$$y = \sum_i \text{diag}(\beta^i)U_{f_{0,i}} \otimes \text{diag}(\alpha^{1,i})U_{f_{1,i}} \otimes \cdots \otimes \text{diag}(\alpha^{n,i})U_{f_{n,i}}.$$

Consider the following element of $C_{n+1}^{bar}(I_{S(\mathfrak{A})})$

$$w = \text{diag}(\gamma) \otimes \text{diag}(\delta)y.$$

We have

$$b'(w) = z - \text{diag}(\gamma) \otimes \text{diag}(\delta)b'(y).$$

If $n = 0$ then $b'(y) = 0$, so this proves that z is a boundary. We have to show that $\text{diag}(\delta)b'(y) = 0$ if $n \geq 1$. Choose a basis $\{v_l\}$ of the \mathbb{Q} -vector space $C_{n-1}^{bar}(I_{S(\mathfrak{A})})$. Then $y = \sum_l T_l \otimes v_l$ for unique $T_l \in I_{S(\mathfrak{A})}$, and

$$0 = b'(z) = \text{diag}(\gamma\delta)b'(y) = \sum_l \text{diag}(\gamma\delta)T_l \otimes v_l.$$

Hence we must have $\text{diag}(\gamma\delta)T_l = 0$ for all l , and therefore $\text{diag}(\delta)b'(y) = 0$ by (8.3.3). \square

Example 8.3.4. Any Banach algebra with a bounded left approximate unit satisfies the Cohen-Hewitt factorization property; thus it has the left TFP ([6, Lemma 6.5.1]). In particular, this applies to C^* -algebras and therefore also to local C^* -algebras. If \mathfrak{A} is a local C^* -algebra then $c_0(\mathfrak{A})$ is again a local C^* -algebra; hence $I_{c_0(\mathfrak{A})}$ is K -excisive, by Proposition 8.3.1.

Example 8.3.5. If \mathfrak{A} is a unital Banach algebra then $\ell^{\infty-}(\mathfrak{A})$ has the TFP. To see this, let $\alpha^1, \dots, \alpha^m \in \ell^{\infty-}$. Choose p such that $\alpha^i \in \ell^p(\mathfrak{A})$ for all i . For each n put

$$\gamma_n = \max_{1 \leq i \leq m} \|\alpha_n^i\|, \quad \beta_n^i = \begin{cases} \alpha_n^i / \gamma_n^{1/2} & \text{if } \gamma_n \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\beta_n^i\| \leq \|\alpha_n^i\|^{1/2}$ and therefore $\beta^i \in \ell^{2p}(\mathfrak{A})$. Similarly $\gamma^{1/4} \in \ell^{4p}(\mathfrak{A})$. One checks that the factorization $\alpha^i = \gamma^{1/4} \gamma^{1/4} \beta^i$ satisfies the requirements of the TFP.

9. HOMOLOGY OF CROSSED PRODUCTS WITH Γ

Throughout this section A and B will be unital \mathbb{Q} -algebras; furthermore, B will be an A -algebra, that is, B will be a \mathbb{Q} -algebra together with a unital \mathbb{Q} -algebra homomorphism $\iota : A \rightarrow B$. Undecorated tensor products are taken over \mathbb{Z} , or equivalently over \mathbb{Q} , since all groups appearing in this section will be \mathbb{Q} -vector spaces.

9.1. Homology of augmented algebras. Assume A is equipped with a left B -module structure and a surjective B -module homomorphism $\pi : B \rightarrow A$ such that $\pi \iota = id_A$. Observe that the triple (B, A, π) is an augmented ring in the sense of Cartan-Eilenberg [3, Chapter VIII, §1]. Since in addition, B is an A -algebra, we call the triple (B, A, π) an *augmented algebra*. Let M be a right B -module. Consider the simplicial A -module $\perp(B/A, M)$ given in dimension n by

$$\perp_n(B/A, M) = M \otimes_A B^{\otimes n},$$

with face and degeneracy maps defined as follows ($n \geq 0$)

$$\begin{aligned} \partial_i : \perp_{n+1}(B/A, M) &\rightarrow \perp_n(B/A, M), \\ \partial_i(x_0 \otimes \cdots \otimes x_{n+1}) &= \begin{cases} x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} & i \leq n \\ x_0 \otimes \cdots \otimes x_n \pi(x_{n+1}) & i = n+1 \end{cases} \\ \delta_i : \perp_n(B/A, M) &\rightarrow \perp_{n+1}(B/A, M), \quad (0 \leq i \leq n) \\ \delta_i(x_0 \otimes \cdots \otimes x_n) &= x_0 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_n. \end{aligned}$$

The homology of $(B/A, M)$ relative to (A, B, π) , denoted $H_*(B/A, M)$, is the homotopy of the simplicial module $\perp(B/A, M)$;

$$H_*(B/A, M) = \pi_*(\perp(B/A, M)) = H_*(\perp(B/A, M), \partial).$$

Here

$$\partial = \sum_{i=0}^{n+1} (-1)^i \partial_i : \perp_{n+1}(B/A, M) \rightarrow \perp_n(B/A, M)$$

is the alternating sum of the face maps. Let $P(B/A) = \perp(B/A, B)$; $\pi : P(B/A) \rightarrow A$ is a resolution which is projective relative to B/A , and $\perp(B/A, N) = N \otimes_B P(B/A)$. Hence if B is flat both as a left and as a right A -module, then

$$H_*(B/A, M) = \text{Tor}_*^B(M, A).$$

Lemma 9.1.1. *Let N be a right B -module. Consider $N^2 = N^{1 \times 2}$ as a right module over $M_2 B$ via the matrix product. View $M_2 B$ as an $A \oplus A$ -algebra*

through the diagonal embedding $(a_1, a_2) \mapsto E_{11}a_1 + E_{22}a_2$. Then the map

$$\begin{aligned} \iota : \perp (B/A, N) &\rightarrow \perp (M_2(B)/A \oplus A, N \oplus N) \\ \iota(x_0 \otimes \cdots \otimes x_n) &= E_{11}x_0 \otimes \cdots \otimes E_{11}x_n \end{aligned}$$

is a quasi-isomorphism.

Proof. Consider the maps

$$\begin{aligned} \iota' : P(B/A)^{2 \times 1} &\rightarrow P(M_2B/A^2), \\ \iota'(E_{i1}(x_0 \otimes \cdots \otimes x_n)) &= E_{i1}x_0 \otimes E_{11}x_1 \otimes \cdots \otimes E_{11}x_n, \\ \text{and } p' : P(M_2B/A^2) &\rightarrow P(B/A)^{2 \times 1}, \\ p'(E_{i_0, i_1}x_0 \otimes \cdots \otimes E_{i_n, i_{n+1}}x_n) &= E_{i_0 1}(x_0 \otimes \cdots \otimes x_n). \end{aligned}$$

One checks that both ι' and p' are M_2B -linear chain homomorphisms, and that $p'\iota' = 1$. In particular $\pi^{2 \times 1} : P(B/A)^{2 \times 1} \rightarrow A^{2 \times 1}$ is a projective resolution relative to M_2A/A^2 , whence

$$\iota = N^{1 \times 2} \otimes_{M_2B} \iota'$$

is a quasi-isomorphism, as claimed. \square

9.2. The augmented algebra $(\Gamma, \mathcal{P}, \epsilon_l)$. Regarding the elements of $2^{\mathbb{N}}$ as sequences of zeros and ones, there is an obvious action of Emb on $2^{\mathbb{N}}$, which agrees with the inner action $f_*(p) = fpf^\dagger$. Thus $\mathbb{Q}[2^{\mathbb{N}}]$ is a $\mathbb{Q}[\text{Emb}]$ -module. Note that, if $A, B \subset \mathbb{N}$ are disjoint, then for $I \subset \mathbb{Q}[2^{\mathbb{N}}]$ as in (5.2) and $q \in 2^{\mathbb{N}}$, we have

$$\begin{aligned} f_*((p_{A \sqcup B} - p_A - p_B)q) &= \\ &= \left(p_{f((A \sqcup B) \cap \text{dom}(f))} - p_{f(A \cap \text{dom}(f))} - p_{f(B \cap \text{dom}(f))} \right) f_*(q) \in I, \\ (f(p_{A \sqcup B} - p_A - p_B)g)_*(q) &= f_*((p_{A \sqcup B} - p_A - p_B)_*(g_*(q))) \\ &= f_*((p_{A \sqcup B} - p_A - p_B) \cdot g_*(q)) \in I. \end{aligned}$$

Thus \mathcal{P} is a Γ -module. Let $f \in \text{Emb}$; put

$$\epsilon_l(f) = p_{\text{ran}(f)} \in 2^{\mathbb{N}} \ni \epsilon_r(f) = \epsilon_l(f^\dagger) = p_{\text{dom}(f)}.$$

Note that

$$\epsilon_l(fg)(n) = p_{\text{ran}(fg)}(n) = \begin{cases} 1 & \text{if } n \in f(\text{dom}(f) \cap \text{ran}(g)) \\ 0 & \text{otherwise} \end{cases} = f_*(\epsilon_l(g))(n).$$

Thus the induced linear map $\epsilon_l : \mathbb{Q}[\text{Emb}] \rightarrow \mathbb{Q}[2^{\mathbb{N}}]$ is a homomorphism of left $\mathbb{Q}[\text{Emb}]$ -modules. In particular, if $A, B \subset \mathbb{N}$ are disjoint, we have

$$\epsilon_l(f(p_{A \sqcup B} - p_A - p_B)g) = f_*(p_{A \sqcup B} - p_A - p_B)\epsilon_l(g) \in I.$$

Hence ϵ_l induces a homomorphism of left Γ -modules

$$\epsilon_l : \Gamma \rightarrow \mathcal{P}.$$

Observe that the canonical inclusion $\mathcal{P} \subset \Gamma$, which is an algebra homomorphism, but not a Γ -module homomorphism, is a section of ϵ_l . Thus we are in the augmented algebra setting described above. By Corollary 6.3.2, every \mathcal{P} -module is flat. In particular Γ is flat over \mathcal{P} and therefore

$$H_*(\Gamma/\mathcal{P}, M) = \mathrm{Tor}_*^\Gamma(M, \mathcal{P}). \quad (9.2.1)$$

In the next lemma and below we consider the following submonoids of Emb

$$\mathrm{Emb} \supset \mathcal{E} = \{f : \mathrm{dom} f = \mathbb{N}\} \supset \mathcal{E}^* = \{f \in \mathcal{E} : \mathrm{ran}(f) = \mathbb{N}\}.$$

If M is a Γ -module and $\mathfrak{S} \in \{\mathcal{E}, \mathcal{E}^*\}$ we write

$$M_{\mathfrak{S}} = M / \mathrm{span}\{m - f_*(m) : f \in \mathfrak{S}\}.$$

Here the span is \mathbb{Z} -linear.

Lemma 9.2.2. *The kernel of $\epsilon_l : \Gamma \rightarrow \mathcal{P}$ is generated, as a left \mathcal{P} -module, by the elements $U_f - 1$, $f \in \mathcal{E}^*$.*

Proof. Let $K = \ker(\epsilon_l)$. It is clear that K is generated, as an abelian group, by the elements $U_f - p_{\mathrm{ran} f}$, $f \in \mathrm{Emb}$. Assume that $f \in \mathrm{Emb}$ but $f \notin \mathcal{E}^*$. We claim that we may choose a subset $A \subset \mathrm{dom}(f)$ such that $B = \mathbb{N} \setminus A$ is bijectable to $\mathbb{N} \setminus f(A)$, and such that $\mathbb{N} \setminus (\mathrm{dom} f \cap B)$ is bijectable to $\mathbb{N} \setminus f(\mathrm{dom} f \cap B)$. Indeed if $\mathbb{N} \setminus \mathrm{dom} f$ is already bijectable to $\mathbb{N} \setminus \mathrm{ran} f$, we may take $A = \mathrm{dom} f$. Otherwise $\mathrm{dom} f$ is infinite, so we may split it into two disjoint infinite pieces, and take A to be one of them. Thus the claim is proved. For such A , there exist $g, h \in \mathcal{E}^*$ such that $g|_A = f|_A$ and $h|_{\mathrm{dom}(f) \cap B} = f|_{\mathrm{dom}(f) \cap B}$. We have

$$p_{\mathrm{ran} f} = p_{f(A)} + p_{f(\mathrm{dom} f \cap B)} \quad \text{and}$$

$$U_f = p_{f(A)} U_{f|_A} + p_{f(\mathrm{dom}(f) \cap B)} U_{f|_{\mathrm{dom}(f) \cap B}} = p_{f(A)} U_g + p_{f(\mathrm{dom}(f) \cap B)} U_h.$$

Thus

$$U_f - p_{\mathrm{ran} f} = p_{f(A)}(U_g - 1) + p_{f(\mathrm{dom} f \cap B)}(U_h - 1).$$

□

Proposition 9.2.3. *Let M be a Γ -module. Then*

$$H_0(\Gamma/\mathcal{P}, M) = M_{\mathcal{E}} = M_{\mathcal{E}^*}.$$

Proof. Immediate from Lemma 9.2.2. □

9.3. Hochschild homology. We recall the basic definitions for Hochschild homology of algebras over a noncommutative base ring ([21, §1.2.11]). If N is a $B \otimes B^{op}$ -module, we write

$$[b, x] = bx - xb \quad (b \in B, x \in N),$$

$$[B, N] = \left\{ \sum_{i=1}^n [b_i, x_i] : b_i \in B, x_i \in N, n \geq 1 \right\},$$

$$N_B = N / [B, N].$$

Next let $A \rightarrow B$ be a unital \mathbb{Q} -algebra homomorphism. Recall from [21, §1.2.11] that the *Hochschild* homology of B relative to A with coefficients in N , denoted $HH_*(B/A, N)$, is the homotopy of the simplicial \mathbb{Q} -module $C(B/A, M)$, which is given in dimension n by

$$C_n(B/A, N) = (N \otimes_A B^{\otimes_A n})_A,$$

with the following face and degeneracy maps

$$\begin{aligned} \mu_i : C_{n+1}(B/A, N) &\rightarrow C_n(B/A, N), \\ \mu_i(x_0 \otimes \cdots \otimes x_{n+1}) &= \begin{cases} x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} & i \leq n \\ x_{n+1} x_0 \otimes \cdots \otimes x_n & i = n+1 \end{cases} \\ s_i : C_n(B/A, N) &\rightarrow C_{n+1}(B/A, N), \quad (0 \leq i \leq n) \\ s_i(x_0 \otimes \cdots \otimes x_n) &= x_0 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_n. \end{aligned}$$

We write b for the alternating sum of the face maps, and $HH(B/A, N)$ for the resulting chain complex. Thus

$$HH_*(B/A, N) = H_*(HH(B/A, N))$$

is the Hochschild homology of B/A with coefficients N . If A is commutative and B is central as an A -bimodule, then $B \otimes_A B^{op}$ is a ring. If furthermore, B happens to be flat as a left A -module, then

$$HH_*(B/A, N) = \mathrm{Tor}_*^{B \otimes_A B^{op}}(B, N).$$

Note this is the case, for example, if $A = \mathbb{Q}$. We shall write $HH_*(B, N)$ for $HH_*(B/\mathbb{Q}, N)$.

Remark 9.3.1. If A and B are commutative and M is a central bimodule, then $C(B/A, M) = M \otimes_B C(B/A, B)$.

Lemma 9.3.2. (cf. [21, Theorem 1.12.13]) *Let $A \rightarrow B$ be a homomorphism of unital \mathbb{Q} -algebras. Assume that A is a filtering colimit of separable \mathbb{Q} -algebras. Then*

$$HH_*(B, N) = HH_*(B/A, N).$$

Proof. It suffices to show that $B \otimes_A B^{op}$ is flat as a $B \otimes B^{op}$ -module. By hypothesis $A = \mathrm{colim}_i A_i$ is a filtering colimit of separable algebras. Hence $B \otimes_A B^{op} = \mathrm{colim}_i B \otimes_{A_i} B^{op}$, so it suffices to prove that if $\mathbb{Q} \subset A$ is separable then $B \otimes_A B$ is flat over $B \otimes_{\mathbb{Q}} B^{op}$, and this is well-known. \square

Example 9.3.3. If A is a \mathcal{P} -algebra, and N an $A \otimes A^{op}$ -module, then $HH_*(A, N) = H_*(A/\mathcal{P}, N)$, by Proposition 6.3.1 and Lemma 9.3.2.

9.4. Hochschild homology of crossed products with Γ . In this subsection, as in (5.9), R will be an Emb-bundle. We also fix an R -bimodule M , central as \mathcal{P} -bimodule, together with a left action of Emb

$$\mathrm{Emb} \times M \rightarrow M, \quad (f, m) \mapsto f_*(m).$$

We require that this action induce a Γ -module structure on M which is *covariant* in the sense that

$$f_*(rms) = f_*(r)f_*(m)f_*(s) \quad (r, s \in R, m \in M). \quad (9.4.1)$$

In this situation, we can form the crossed product $M \#_{\mathcal{P}} \Gamma$; this is the $R \#_{\mathcal{P}} \Gamma$ -bimodule consisting of $M \otimes_{\mathcal{P}} \Gamma$ equipped with the following left and right actions of $R \#_{\mathcal{P}} \Gamma$

$$(a \# U_f)(m \# U_g) = af_*(m) \# U_{fg}, \quad (m \# U_g)(a \# U_f) = mg_*(a) \# U_{gf}.$$

We are interested in the Hochschild homology of $R \#_{\mathcal{P}} \Gamma$ with coefficients in $M \#_{\mathcal{P}} \Gamma$, which by Example 9.3.3 is computed by the simplicial \mathcal{P} -module $C(R \#_{\mathcal{P}} \Gamma / \mathcal{P}, M \#_{\mathcal{P}} \Gamma)$. On the other hand it is not hard to check, using (9.4.1) and the definition of Emb-bundle, that the diagonal action of Emb on $C(R)$ descends to an action of Γ on $C(R/\mathcal{P})$. Hence we may also consider the bisimplicial module $\perp (\Gamma/\mathcal{P}, C(R/\mathcal{P}, M))$ which results from applying the functor $\perp (\Gamma/\mathcal{P}, -)$ dimension-wise to the simplicial module $C(R/\mathcal{P}, M)$. The diagonal of this bisimplicial module is

$$\begin{aligned} \text{diag}(\perp (\Gamma/\mathcal{P}, C(R/\mathcal{P}, M)))_n = \\ \perp^n (\Gamma/\mathcal{P}, C_n(R, M)) = (M \otimes_{\mathcal{P}} R^{\otimes_{\mathcal{P}} n})_{\mathcal{P}} \otimes_{\mathcal{P}} \Gamma^{\otimes_{\mathcal{P}} n}, \end{aligned}$$

with faces $\mu_i \partial_i$ and degeneracies $s_i \delta_i$. The simplicial module

$$\text{diag}(\perp (\Gamma/\mathcal{P}, C(R/\mathcal{P}, M)))$$

is a model for the hyperhomology of Γ/\mathcal{P} with coefficients in $C(R/\mathcal{P}, M)$; hence, if $\mathbb{H}(\Gamma/\mathcal{P}, C(R/\mathcal{P}, M))$ is any other such model, we have a quasi-isomorphism

$$\mathbb{H}(\Gamma/\mathcal{P}, C(R/\mathcal{P}, M)) \xrightarrow{\sim} \text{diag}(\perp (\Gamma/\mathcal{P}, C(R/\mathcal{P}, M))).$$

Observe that any element of $\text{diag}(\perp (\Gamma/\mathcal{P}, C(R/\mathcal{P}, M)))_n$ can be written as a sum of congruence classes of elementary tensors of the form

$$x = a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes f_1 \otimes \cdots \otimes f_n, \quad (9.4.2)$$

where $a_0 \in M$, $a_i \in R$, and $f_i \in \text{Emb}$ ($i \geq 1$) are such that

$$\begin{aligned} \epsilon_l(f_i^\dagger) &= \epsilon_l(f_{i+1}) \quad (1 \leq i \leq n-1), \\ a_j \epsilon_l(f_1) &= a_j \quad (0 \leq j \leq n). \end{aligned}$$

Next we define a map

$$\phi : \text{diag}(\perp (\Gamma/\mathcal{P}, C(R/\mathcal{P}, M))) \rightarrow C_n(R \#_{\mathcal{P}} \Gamma / \mathcal{P}, M \#_{\mathcal{P}} \Gamma).$$

For x as in (9.4.2), we put

$$\phi([x]) = [a_0 \# f_1 \otimes f_1^\dagger(a_1) \# f_2 \otimes \cdots \otimes (f_1 \cdots f_n)^\dagger(a_n) \# (f_1 \cdots f_n)^\dagger]. \quad (9.4.3)$$

Here $[]$ denotes congruence class.

Proposition 9.4.4. *The assignment (9.4.3) gives a simplicial isomorphism*

$$\phi : \text{diag}(\perp (\Gamma/\mathcal{P}, C(R/\mathcal{P}, M))) \xrightarrow{\cong} C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}, M\#_{\mathcal{P}}\Gamma).$$

In particular, we have a quasi-isomorphism

$$\mathbb{H}(\Gamma/\mathcal{P}, HH(R/\mathcal{P}, M)) \xrightarrow{\sim} HH(R\#_{\mathcal{P}}\Gamma/\mathcal{P}, M\#_{\mathcal{P}}\Gamma).$$

Proof. First of all, we must check that (9.4.3) gives a well-defined simplicial homomorphism. To do this, one checks first that formula (9.4.3) defines a simplicial homomorphism

$$\hat{\phi} : \text{diag}(\perp (\mathbb{Q}[\text{Emb}], C(R, M))) \rightarrow C(R\#\text{Emb}, M\#\text{Emb}).$$

Then one observes that it passes down to the quotient, inducing a map $\phi : \text{diag}(\perp (\Gamma/\mathcal{P}, C(R, M))) \rightarrow C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}, M\#_{\mathcal{P}}\Gamma)$. Next note that the image of $\hat{\phi}$ is contained in the simplicial subgroup $S \subset C(R\#\text{Emb}, M\#\text{Emb})$ given in dimension n by

$$S_n = \text{span}\{[a_0\#f_0 \otimes \cdots \otimes a_n\#f_n] : f_i \in \text{Emb}, \ a_i \in R, \ f_0 \cdots f_n \in 2^{\mathbb{N}}\}.$$

To prove that ϕ is surjective, we must show that $S \rightarrow C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}, M\#_{\mathcal{P}}\Gamma)$ is surjective. Any element of $C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}, M\#_{\mathcal{P}}\Gamma)$ can be written as a linear combination of classes of elementary tensors of the form

$$y = a_0\#f_0 \otimes \cdots \otimes a_n\#f_n, \quad (9.4.5)$$

such that the following conditions are satisfied for $0 \leq i \leq n-1$ and $0 \leq j \leq n$:

$$\epsilon_r(f_i) = \epsilon_l(f_{i+1}), \quad \epsilon_r(f_n) = \epsilon_l(f_0) \quad a_j = a_j \epsilon_l(f_j). \quad (9.4.6)$$

Let $f = f_0 \cdots f_n$; then $\text{dom}(f) = \text{ran}(f) = \text{ran}(f_0) = \text{dom}(f_n)$. Let

$$\mathbb{N} \supset A = \{x \in \text{dom}(f) : f(x) = x\}.$$

If $A = \text{dom}(f)$ then $f \in 2^{\mathbb{N}}$, and thus the element (9.4.5) belongs to S . Otherwise, by Zorn's Lemma, there exists $\emptyset \neq B \subset \text{dom}(f)$ maximal with the property that $f(B) \cap B = \emptyset$. Clearly $A \cap B = \emptyset$; let $C = \text{dom}(f) \setminus (A \sqcup B)$. Then $f(B) \subset C$, $f(C) \subset B$, and $p_{\text{dom}(f)} = p_A + p_B + p_C$. Hence we have

$$[y] = [p_{\text{dom}(f)} y p_{\text{dom}(f)}] = [p_A y p_A] = [a_0\#g_0 \otimes \cdots \otimes a_n\#g_n],$$

for $g_n = (f_n)|_A$ and $g_i = (f_i)|_{f_{i+1} \cdots f_n(A)}$ ($0 \leq i \leq n-1$). In particular $g_0 \cdots g_n = p_A$. Thus ϕ is surjective. To prove it is injective, define a map

$$\psi : C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}, M\#_{\mathcal{P}}\Gamma) \rightarrow \text{diag}(\perp (\Gamma/\mathcal{P}, C(R, M)))$$

as follows. For y as in (9.4.5) satisfying the conditions (9.4.6) and such that $f_0 \cdots f_n \in 2^{\mathbb{N}}$, put

$$\psi([y]) = [a_0 \otimes f_0(a_1) \otimes \cdots \otimes (f_0 \cdots f_{n-1})(a_n) \otimes f_0 \otimes \cdots \otimes f_{n-1}].$$

One checks that ψ is well-defined and that $\psi\phi = \text{id}$. \square

Corollary 9.4.7. *Assume that R is commutative and that M is a central R -bimodule. Then*

$$HH_0(R \#_{\mathcal{P}} \Gamma, M \#_{\mathcal{P}} \Gamma) = M_{\mathcal{E}}.$$

Proof. By Proposition 9.4.4,

$$HH_0(R \#_{\mathcal{P}} \Gamma, M \#_{\mathcal{P}} \Gamma) = H_0(\Gamma/\mathcal{P}, HH_0(R, M)).$$

By our assumptions on R and M , $HH_0(R, M) = M$. Finally we have $H_0(\Gamma/\mathcal{P}, M) = M_{\mathcal{E}}$, by Proposition 9.2.3. \square

9.5. Comparing the 0^{th} -homology of (Γ^{∞}, I_S) and that of $(\mathcal{B} : J_S)$.

Proposition 9.5.1. *Let $S \triangleleft \ell^{\infty}$ be a symmetric ideal and let $J_S \triangleleft \mathcal{B} = \mathcal{B}(\ell^2)$ be the corresponding ideal of bounded operators in ℓ^2 . Then the inclusion $\Gamma^{\infty} \subset \mathcal{B}$ induces an isomorphism*

$$HH_0(\Gamma^{\infty}, I_S) \xrightarrow{\cong} HH_0(\mathcal{B}, J_S).$$

Proof. By Proposition 5.11 Corollary 9.4.7, the inclusion $\text{diag} : S \rightarrow I_S$ descends to a bijection

$$S_{\mathcal{E}} \xrightarrow{\cong} HH_0(\Gamma^{\infty}, I_S). \quad (9.5.2)$$

By [15, Theorem 5.12] the composite of (9.5.2) with the map induced by the inclusion $I_S \subset J_S$ is an isomorphism. \square

Corollary 9.5.3. *The map $HC_0(\Gamma^{\infty} : I_S) \rightarrow HC_0(\mathcal{B} : J_S)$ is an isomorphism.*

Proof. It follows from Proposition 9.5.1 and the fact that, if R is a unital ring and $I \triangleleft R$ is an ideal then

$$HH_0(R : I) = HC_0(R : I) = I/[R, I].$$

\square

Lemma 9.5.4. *Let $p > 0$. Then:*

$$\begin{aligned} HC_0(\Gamma^{\infty} : I_{\ell^{p+}}) &= \begin{cases} \mathbb{C} & p < 1 \\ 0 & p \geq 1 \end{cases} \\ HC_0(\Gamma^{\infty} : I_{\ell^{p-}}) &= \begin{cases} \mathbb{C} & p \leq 1 \\ 0 & p > 1 \end{cases} \\ HC_0(\Gamma^{\infty} : I_{\ell^p}) &= \begin{cases} \mathbb{C} & p < 1 \\ \mathbb{C} \oplus \mathbb{V} & p = 1 \\ 0 & p > 1. \end{cases} \end{aligned}$$

Here \mathbb{V} is a \mathbb{C} -vector space of uncountable dimension.

Proof. It follows from Corollary 9.5.3 and [32, pages 492-493]. \square

9.6. Cyclic homology of $R\#_{\mathcal{P}}\Gamma$. Let M be a right Γ -module. Consider the simplicial module $\perp(\Gamma/\mathcal{P}, M)$. Every element of $\perp_n(\Gamma/\mathcal{P}, M)$ can be written as a sum of elementary tensors

$$x = m \otimes f_1 \otimes \cdots \otimes f_n$$

with $m \in M$, $f_i \in \text{Emb}$, and $\text{dom}(f_i) = \text{ran}(f_{i+1})$ ($i < n$). For x as above, put

$$\tau_n(x) = (-1)^n m(f_1 \cdots f_n) \otimes (f_1 \cdots f_n)^\dagger \otimes f_1 \otimes \cdots \otimes f_{n-1}. \quad (9.6.1)$$

One checks that the assignment (9.6.1) gives a well-defined endomorphism of $\perp_n(\Gamma/\mathcal{P}, M)$, and that the cyclic identities [21, 2.5.1.1] hold. Thus the simplicial (\mathbb{Q} -)module $\perp(\Gamma/\mathcal{P}, M)$, equipped with the cyclic operators τ_n ($n \geq 0$), is a *cyclic module*. In general if \mathcal{C} is any cyclic module, then we can equip \mathcal{C} with a map $B : \mathcal{C} \rightarrow \mathcal{C}[+1]$ called the Connes' operator, which, together with the usual boundary $b : \mathcal{C} \rightarrow \mathcal{C}[-1]$ given by the alternating sum of the face maps, satisfy $b^2 = B^2 = [b, B] = 0$. The *Hochschild complex* of \mathcal{C} is $HH(\mathcal{C}) = (\mathcal{C}, b)$. The *cyclic* and *negative cyclic* complexes are the complexes given in dimension n by $HC(\mathcal{C})_n = \bigoplus_{m \geq 0} \mathcal{C}_{n-2m}$ and $HN(\mathcal{C})_n = \prod_{m \geq 0} \mathcal{C}_{n+2m}$; they are equipped with the boundary $b + B$. Observe that $HC(\mathcal{C})$ is also equipped with a chain map $S : HC(\mathcal{C}) \rightarrow HC(\mathcal{C})[-2]$ defined by the obvious projections $HC(\mathcal{C})_n \rightarrow HC(\mathcal{C})_{n-2}$. If C is another chain complex equipped with a chain map $S : C \rightarrow C[-2]$, then by a *map of S -complexes* $C \rightarrow HC(\mathcal{C})$ we understand a chain map which commutes with S . When $\mathcal{C} = \perp(\Gamma/\mathcal{P}, M)$, we write ∂ and \mathcal{B} for the operators b and B .

Proposition 9.6.2. *There is a natural quasi-isomorphism of S -complexes $(HC(\perp(\Gamma/\mathcal{P}, M)), \partial) \rightarrow (HC(\perp(\Gamma/\mathcal{P}, M)), \partial + \mathcal{B})$.*

Proof. View $\mathcal{C} = \perp(\Gamma/\mathcal{P}, M)$ as a cyclic module. Consider the projection

$$\pi : HN(\mathcal{C})_n = \prod_{m \geq 0} \mathcal{C}_{n+2m} \rightarrow \mathcal{C}_n = HH(\mathcal{C})_n.$$

Observe that $\pi(b + \mathcal{B}) = b\pi$. Proceed as in [12, §3.1] to define a chain map $\Upsilon : HH(\mathcal{C}) \rightarrow HN(\mathcal{C})$ such that $\pi\Upsilon = 1$. We have a chain map $\theta^n : HN(\mathcal{C}) \rightarrow HC(\mathcal{C})[2n]$ ($n \geq 0$) given by the composite

$$\begin{aligned} \theta^n : HN(\mathcal{C})_p &= \prod_{m \geq 0} \mathcal{C}_{p+2m} \rightarrow \bigoplus_{m=0}^n \mathcal{C}_{p+2m} \\ &\subset \bigoplus_{q \geq 0} \mathcal{C}_{p+2(n-q)} = HC(\mathcal{C})_{p+2n}. \end{aligned}$$

The map of the proposition is

$$\sum_{n=0}^{\infty} \theta^n \Upsilon : (HC(\mathcal{C}), \partial) = \bigoplus_{n \geq 0} HH(\mathcal{C})[-2n] \rightarrow (HC(\mathcal{C}), b + \mathcal{B}).$$

□

Theorem 9.6.3. *Let R be an Emb-bundle. There is a natural zig-zag of quasi-isomorphisms*

$$\mathbb{H}(\Gamma/\mathcal{P}, HC(R/\mathcal{P})) \xrightarrow{\sim} HC(R\#\Gamma).$$

Proof. Consider the bicyclic module

$$\mathcal{C}_{*,*} : ([m], [n]) \mapsto \perp_m (\Gamma/\mathcal{P}, C_n(R/\mathcal{P})). \quad (9.6.4)$$

It follows from Proposition 9.6.2 that the total cyclic complex

$$T = (HC(\mathcal{C}_{*,*}), b + \partial + B + \mathcal{B})$$

is quasi-isomorphic to

$$(HC(\mathcal{C}_{*,*}), b + \partial + B),$$

which in turn is a model for $\mathbb{H}(\Gamma/\mathcal{P}, HC(R/\mathcal{P}))$. By the cylindrical version of the Eilenberg-Zilber theorem ([20, Theorem 3.1]), the complex T is S -equivalent to the HC -complex of the diagonal Δ of (9.6.4). By Proposition (9.4.4), the map (9.4.3) is an isomorphism of simplicial modules $\Delta \xrightarrow{\cong} C(R\#\Gamma/\mathcal{P})$; one checks that it is actually an isomorphism of cyclic modules. Finally, by Example 9.3.3, the projection $C(R\#\Gamma) \rightarrow C(R\#\Gamma/\mathcal{P})$ induces a quasi-isomorphism $HC(R\#\Gamma) \rightarrow HC(R\#\Gamma/\mathcal{P})$. \square

9.7. Hodge decomposition. If R is a commutative \mathbb{Q} -algebra, then there are defined Adams operations on $C(R)$, and we have an eigenspace decomposition [21, Theorems 4.5.10 and 4.6.7]

$$C(R) = \bigoplus_{p \geq 0} C^{(p)}(R), \quad (9.7.1)$$

called the *Hodge decomposition*. We have $C_n^{(p)} = 0$ for $n < p$ and each $C^{(p)}$ is a graded R -submodule, closed under the Hochschild boundary map b . Thus, if M is a central R -bimodule, for $HH^{(p)}(R, M) = M \otimes_R (C^{(p)}(R), b)$ we have

$$HH_n(R, M) = \bigoplus_{p \geq 0}^n HH_n^{(p)}(R, M).$$

The Connes operator B sends $C^{(p)}$ to $C^{(p+1)}$. Thus, we have a direct sum decomposition of the cyclic complex

$$HC(R) = \bigoplus_{p=0}^{\infty} HC^{(p)}(R)$$

where

$$HC^{(p)}(R)_n = \bigoplus_{p \geq 0}^n C_{n-2p}^{(n-p)}(R).$$

Hence for $HC_*^{(p)}(R) = H_*(HC^{(p)}(R))$,

$$HC_n(R) = \bigoplus_{p=0}^n HC_n^{(p)}(R).$$

Let (Ω_R^*, d) for the DGA of (absolute) Kähler differential forms. There is a natural map of mixed complexes

$$\begin{aligned} \mu : (C(R), b, B) &\rightarrow (\Omega_R, 0, d) \\ \mu(x_0 \otimes \cdots \otimes x_n) &= (1/n!)x_0 dx_1 \wedge \cdots \wedge dx_n. \end{aligned} \quad (9.7.2)$$

Let M be a central R -bimodule; the map μ induces isomorphisms

$$HH_n^{(n)}(R, M) = M \otimes_R \Omega_R^n \quad (9.7.3)$$

$$\text{and } HC_n^{(n)}(R) = \Omega_R^n / d(\Omega_R^{n-1}). \quad (9.7.4)$$

We say that R is *homologically smooth* if (9.7.2) is a quasi-isomorphism.

Example 9.7.5. Let R be a unital commutative complex C^* -algebra over \mathbb{C} . It was proved in [11, Thm. 8.2.6] that R , regarded as a \mathbb{Q} -algebra, is homologically smooth. In particular this applies when $R = \ell^\infty$. Moreover, by [11, proof of Prop. 5.2.2], ℓ^∞ is a filtering colimit of smooth \mathbb{C} -algebras. It follows that $\Omega_{\ell^\infty}^n$ is a flat ℓ^∞ -module for every n . Hence

$$HH_n(\ell^\infty, M) = M \otimes_{\ell^\infty} \Omega_{\ell^\infty}^n$$

for every central bimodule M .

Now assume that the commutative algebra R is an Emb-bundle. Then by Proposition 9.4.4, Theorem 9.6.3, and naturality of the Hodge decomposition, we have quasi-isomorphisms

$$HH(R \# \Gamma, M \# \Gamma) \xrightarrow{\sim} \bigoplus_{p \geq 0} \mathbb{H}(\Gamma/\mathcal{P}, HH^{(p)}(R, M)) \quad (9.7.6)$$

$$\text{and } HC(R \# \Gamma) \xrightarrow{\sim} \bigoplus_{p \geq 0} \mathbb{H}(\Gamma/\mathcal{P}, HC^{(p)}(R/\mathcal{P})). \quad (9.7.7)$$

Put

$$\begin{aligned} HH_n^{(p)}(R \# \Gamma, M \# \Gamma) &= \mathbb{H}_n(\Gamma/\mathcal{P}, HH^{(p)}(R, M)), \\ HC_n^{(p)}(R \# \Gamma, M \# \Gamma) &= \mathbb{H}_n(\Gamma/\mathcal{P}, HC^{(p)}(R, M)). \end{aligned} \quad (9.7.8)$$

We have decompositions

$$\begin{aligned} HH_n(R \# \Gamma, M \# \Gamma) &= \bigoplus_{p=0}^n HH_n^{(p)}(R \# \Gamma, M \# \Gamma), \\ HC_n(R \# \Gamma) &= \bigoplus_{p=0}^n HC_n^{(p)}(R \# \Gamma). \end{aligned}$$

It follows from (9.7.3), (9.7.4), and Proposition 9.2.3 that

$$\begin{aligned} HH_n^{(n)}(R \# \Gamma, M \# \Gamma) &= (M \otimes_R \Omega_R^n)_{\mathcal{E}}, \\ HC_n^{(n)}(R \# \Gamma) &= (\Omega_R^n / d\Omega_R^{n-1})_{\mathcal{E}}. \end{aligned} \quad (9.7.9)$$

10. THE RELATIVE CYCLIC HOMOLOGY $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$

10.1. The Quillen spectral sequence. Let R be a unital \mathbb{Q} -algebra and $I \triangleleft R$ a two-sided ideal, flat as a left ideal. Then

$$I^{\otimes_R^n} \cong I^n$$

Using the isomorphism above and flatness again we see that if $P \xrightarrow{\sim} I$ is a projective bimodule resolution, then $Q = P^{\otimes_R^n} \xrightarrow{\sim} I^n$ is again a resolution. Hence modding out Q by the commutator subspace $[Q, R]$ we obtain a complex which computes $HH_*(R, I^n)$ and which has a natural action of $\mathbb{Z}/n\mathbb{Z}$ via permutation of factors. Following Quillen [23, pp 210] we shall write $HH_*(R, I^n)_\sigma$ for the coinvariants of this action. Quillen introduced a first quadrant spectral sequence (see [23, Proposition 2.16 and Theorem 4.3]),

$$E_{p,q}^1 = \begin{cases} HC_q(R) & p = 0 \\ HH_{q-p+1}(R, I^p)_\sigma & p \geq 1, \end{cases} \quad (10.1.1)$$

which converges to $HC_{p+q}(R/I)$. For example, every ideal $J \triangleleft \mathcal{B} = \mathcal{B}(\ell^2)$ of the algebra of bounded operators is flat; M. Wodzicki has used this spectral sequence, together with the results of [15], to study the relative cyclic homology groups $HC_*(\mathcal{B} : J)$. By Proposition 6.2.5, every ideal of Γ^∞ is flat; by Proposition 6.2.7 and Examples 6.2.4, the same is true of $I_{c_0(\mathfrak{A})}$ and $I_{\ell^\infty-(\mathfrak{A})}$ for every unital Banach algebra \mathfrak{A} . In this subsection we shall use Quillen's spectral sequence to study the cyclic homology groups $HC_*(\Gamma^\infty : I_S)$ for general S , and to show that $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{\ell^\infty-(\mathfrak{A})}) = HC_*(\Gamma^\infty(\mathfrak{A}) : I_{c_0(\mathfrak{A})}) = 0$. Proposition 10.1.5 below will play a role akin to that played by [32, Theorem 8] in the context of operator ideals. Let \mathfrak{A} and \mathfrak{B} be Banach algebras. We have maps

$$\Gamma \otimes \Gamma \rightarrow \Gamma(\mathbb{N} \times \mathbb{N}), \quad U_f \otimes U_g \mapsto U_{f \times g}, \quad (10.1.2)$$

$$\boxtimes : \ell^\infty(\mathfrak{A}) \otimes \ell^\infty(\mathfrak{B}) \rightarrow \ell^\infty(\mathbb{N} \times \mathbb{N}, \mathfrak{A} \hat{\otimes} \mathfrak{B}), \quad (\alpha \boxtimes \beta)_{m,n} = \alpha_n \hat{\otimes} \beta_m. \quad (10.1.3)$$

These two maps together induce

$$\begin{aligned} \Gamma^\infty(\mathfrak{A}) \otimes \Gamma^\infty(\mathfrak{B}) &\rightarrow \\ \Gamma^\infty(\mathbb{N} \times \mathbb{N}, \mathfrak{A} \hat{\otimes} \mathfrak{B}) &:= \ell^\infty(\mathbb{N} \times \mathbb{N}, \mathfrak{A} \hat{\otimes} \mathfrak{B}) \#_{\mathcal{P}(\mathbb{N} \times \mathbb{N})} \Gamma(\mathbb{N} \times \mathbb{N}). \end{aligned}$$

We write $\Gamma^\infty(\mathbb{N} \times \mathbb{N}) = \Gamma^\infty(\mathbb{N} \times \mathbb{N}, \mathbb{C})$. In particular we have a map

$$\Gamma^\infty \otimes \Gamma^\infty \rightarrow \Gamma^\infty(\mathbb{N} \times \mathbb{N}). \quad (10.1.4)$$

Proposition 10.1.5. (cf. [32, Theorem 8]) *Let $S, T \triangleleft \ell^\infty$ be symmetric ideals, and let \mathfrak{B} be a unital Banach algebra. Assume that*

- i) *The map (10.1.3) sends $S \otimes T \rightarrow T(\mathbb{N} \times \mathbb{N})$.*
- ii) $S_{\mathcal{E}} = 0$.

Then

$$HH_*(\Gamma^\infty(\mathfrak{B}), I_{T(\mathfrak{B})}) = 0.$$

Proof. Proceeding as in the proof of Proposition 7.3.4, we obtain a commutative diagram

$$\begin{array}{ccc} \Gamma^\infty \otimes \Gamma^\infty(\mathfrak{B}) & \longrightarrow & M_2 \Gamma^\infty(\mathfrak{B}) \\ E_{1,1} \otimes - \uparrow & \nearrow & \\ \Gamma^\infty(\mathfrak{B}) & & \end{array}$$

By hypothesis i) this restricts to a commutative diagram

$$\begin{array}{ccc} I_S \otimes I_{T(\mathfrak{B})} & \longrightarrow & M_2 I_{T(\mathfrak{B})} \\ E_{1,1} \otimes - \uparrow & \nearrow & \\ I_{T(\mathfrak{B})} & & \end{array}$$

Now use hypothesis ii), Morita invariance and the Künneth formula for Hochschild homology ([21, Theorem 1.2.4] and [29, Proposition 9.4.1]), and induction, to conclude that $HH_*(\Gamma^\infty(\mathfrak{A}), I_{T(\mathfrak{A})}) = 0$. \square

We shall need the following result of Dykema, Figiel, Weiss and Wodzicki, which follows by combining [15, Theorem 5.11(ii)] and Theorem 5.12].

Proposition 10.1.6. ([15]) *Let $S \triangleleft \ell^\infty$ be a symmetric ideal and let $\omega = (1/n)_{n \geq 1}$ be the harmonic sequence. Then*

$$S_{\mathcal{E}} = 0 \iff \omega \boxtimes S \subset S(\mathbb{N} \times \mathbb{N}).$$

Proposition 10.1.7.

- i) $HC_*(\Gamma^\infty : I_{c_0}) = HC_*(\mathcal{B} : J_{c_0}) = 0$.
- ii) $HC_*(\Gamma^\infty : I_{\ell^\infty-}) = HC_*(\mathcal{B} : J_{\ell^\infty-}) = 0$.
- iii) Let $0 < p < \infty$, $S \in \{\ell^p, \ell^{p-}, \ell^{p+}\}$,

$$m = \min\{n : HC_n(\Gamma^\infty : I_S) \neq 0\},$$

$$m' = \min\{n : HC_n(\mathcal{B} : J_S) \neq 0\}.$$

Then $m = m'$ and the map $HC_m(\Gamma^\infty : I_S) \rightarrow HC_m(\mathcal{B} : J_S)$ is an isomorphism.

Proof. Consider the spectral sequence (10.1.1) in the cases $R = \Gamma^\infty, \mathcal{B}$ and $I = I_S, J_S$ for each of the symmetric ideals S of the proposition. We have $E_{0,*}^1 = 0$ since both Γ^∞ and \mathcal{B} are rings with infinite sums. In both i) and ii), we have $S^2 = S$ and $\omega \boxtimes S \subset S(\mathbb{N} \times \mathbb{N})$ whence $E_{*,*}^1 = 0$, by Propositions 10.1.6 and 10.1.5 and [32, Theorem 8]. This gives i) and ii). In each of the cases considered in part iii), we have $S \boxtimes S \subset S(\mathbb{N} \times \mathbb{N})$. Since $\omega \in \ell^p$ if and only if $p > 1$ and since $(\ell^p)^n = \ell^{p/n}$, we have $HH_*(\Gamma^\infty, I_{(\ell^p)^n}) = HH_*(\mathcal{B}, (\mathcal{L}^p)^n) = 0$ for $p/n > 1$, again by Propositions 10.1.6 and 10.1.5 and [32, Theorem 8]. The case $S = \ell^p$ follows from this and from Corollary 9.5.1. The remaining cases follow similarly. \square

Remark 10.1.8. Proposition 10.3.3 below provides a more detailed computation of $HC_n(\Gamma^\infty : I_S)$ for S as in case iii) of Proposition 10.1.7 above.

Theorem 10.1.9. *The comparison map $K_*(I_{S(\mathfrak{A})}) \rightarrow KH_*(I_{S(\mathfrak{A})})$ is an isomorphism in the following cases:*

- i) $S = c_0$ and \mathfrak{A} is a C^* -algebra.
- ii) $S = \ell^\infty$ and \mathfrak{A} is a unital Banach algebra.

Proof. By Proposition 8.3.1 and Examples 8.3.4 and 8.3.5, $I_{S(\mathfrak{A})}$ is H -unital in both cases. Hence by Theorem 8.2.1 it suffices to show that $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) = 0$. As explained in the proof of Proposition (10.1.7), Proposition 10.1.6 implies that $S_{\mathcal{E}} = 0$. Hence if \mathfrak{A} is unital we are done by Propositions 6.2.7 and 10.1.5; in particular, part ii) is proved. The nonunital case of i) follows from the unital case using excision. \square

10.2. Computing $HC^{(p)}(\Gamma^\infty : I_S)$ in terms of differential forms. Let $S \triangleleft \ell^\infty$ be an ideal. Consider the subcomplex

$$\mathcal{F}_p(S) \subset \Omega_{\ell^\infty} \quad (10.2.1)$$

$$(\mathcal{F}_p(S))^q = \begin{cases} S^{p-q+1} \Omega_{\ell^\infty}^q & p \geq q \\ \Omega_{\ell^\infty}^q & q > p. \end{cases}$$

Write

$$D^{(p)}(S)_q = (\Omega_{\ell^\infty}^{-q} / (\mathcal{F}_p^{-q}(S))) \quad (10.2.2)$$

$$L^{(p)}(S)_q = \mathcal{F}_{p-1}^{-q}(S) / \mathcal{F}_p^{-q}(S). \quad (10.2.3)$$

Note $L^{(p)}(S)$ and $D^{(p)}(S)$ are nonpositive chain complexes.

Theorem 10.2.4. *Let $S \triangleleft \ell^\infty$ be a symmetric ideal. Then there are Emb-equivariant quasi-isomorphisms*

$$\begin{aligned} HH^{(p)}(\ell^\infty/S) &\xrightarrow{\sim} L^{(p)}(S)[p] \\ HC^{(p)}(\ell^\infty/S) &\xrightarrow{\sim} D^{(p)}(S)[p]. \end{aligned}$$

Proof. Consider the skew-commutative graded algebra $\Lambda = \ell^\infty \oplus S$ with grading $\Lambda_0 = \ell^\infty$, $\Lambda_1 = S$. The inclusion $S \subset \ell^\infty$ defines a homogeneous ℓ^∞ -linear derivation $\partial : \Lambda \rightarrow \Lambda[-1]$. Thus Λ is a chain DGA, and the projection $\ell^\infty \rightarrow \ell^\infty/S$ defines a quasi-isomorphism of cyclic modules $C(\Lambda, \partial) \xrightarrow{\sim} C(\ell^\infty/S)$. By [8, Thms. 2.6 and 3.3] and Proposition 6.2.1, there are quasi-isomorphisms $C(\Lambda, \partial) \xrightarrow{\sim} \bigoplus_p L^{(p)}(S)[p]$ and $\mathfrak{B}(\Lambda, \partial) \xrightarrow{\sim} \bigoplus_p D^{(p)}(S)[p]$; by [27] they are compatible with the Hodge decomposition. Finally, all these quasi-isomorphisms are natural, and thus Emb-equivariant. \square

Theorem 10.2.5.

$$\begin{aligned} HC_*^{(p)}(\Gamma^\infty : I_S) &= \mathbb{H}_{*+p}(\Gamma/\mathcal{P}, \mathcal{F}_{(p)}(S)) \\ HH_*^{(p)}(\Gamma^\infty : I_S) &= \mathbb{H}_{*+p+1}(\Gamma/\mathcal{P}, L_{(p)}(S)). \end{aligned}$$

Proof. It follows from (9.7.8) using Theorem 10.2.4 and the fact that Γ^∞ is an infinite sum ring. \square

Corollary 10.2.6. *There is a first quadrant homological spectral sequence*

$${}_pE_{m,n}^1 = H_n(\Gamma/\mathcal{P}, S^{m+1}\Omega_{\ell^\infty}^{p-m}) \Rightarrow HC_{m+n+p}^{(p)}(\Gamma^\infty : I_S).$$

Proof. This is the spectral sequence associated to $\mathbb{H}(\Gamma/\mathcal{P}, \mathcal{F}_{(p)}(S))$. It is located in the first quadrant because as Γ^∞ is an infinite sum ring,

$$HH_*^{(q)}(\Gamma^\infty) = H_{*+q}(\Gamma/\mathcal{P}, \Omega_{\ell^\infty}^q) = 0.$$

□

Corollary 10.2.7.

$$HC_n^{(n)}(\Gamma^\infty : I_S) = (S\Omega_{\ell^\infty}^n / d(S^2\Omega_{\ell^\infty}^n))_{\mathcal{E}}$$

Proof. It follows from inspection of the second term of the spectral sequence of Corollary 10.2.6, by using the fact that $H_0(\Gamma/\mathcal{P}, -) = (\)_{\mathcal{E}}$ is right exact. □

10.3. The cases $S = \ell^p, \ell^{p\pm}$.

Lemma 10.3.1. *Let $S \triangleleft \ell^\infty$ be a symmetric ideal. Then the map*

$$C(\Gamma/\mathcal{P}, S\Omega_{\ell^\infty}^p) \rightarrow C(\Gamma(\mathbb{N} \sqcup \mathbb{N})/\mathcal{P}(\mathbb{N} \sqcup \mathbb{N}), S(\mathbb{N} \sqcup \mathbb{N})\Omega_{\ell^\infty(\mathbb{N} \sqcup \mathbb{N})}^p)$$

induced by the inclusion $\mathbb{N} \subset \mathbb{N} \sqcup \mathbb{N}$ into the first copy, is a quasi-isomorphism.

Proof. Recall from Example 9.7.5 that $\Omega_{\ell^\infty}^p$ is a flat ℓ^∞ -module. It follows that the map $S \otimes_{\ell^\infty} \Omega_{\ell^\infty}^p \rightarrow S\Omega_{\ell^\infty}^p$ is an isomorphism for every ideal S . Now the proof is immediate from Lemmas 7.3.1 and 9.1.1. □

Lemma 10.3.2. *Let $0 \neq S_1, S_2 \subset \ell^\infty$ be symmetric ideals. Assume that $(S_1)_{\mathcal{E}} = 0$ and that the map $\ell^\infty \otimes \ell^\infty \rightarrow \ell^\infty(\mathbb{N} \times \mathbb{N})$ sends $S_1 \otimes S_2 \rightarrow S_2(\mathbb{N} \times \mathbb{N})$. Then $H_*(\Gamma/\mathcal{P}, S_2\Omega_{\ell^\infty}^p) = 0$ ($p \geq 0$).*

Proof. The proof follows using Lemma 10.3.1 and the argument of the proof of Proposition 10.1.5. □

Let $p > 0$; the following notation is used in the proposition below.

$$\lfloor p \rfloor = \begin{cases} p-1 & p \in \mathbb{Z} \\ \lfloor p \rfloor & p \notin \mathbb{Z}. \end{cases}$$

Note $\lfloor p \rfloor = \lfloor p \rfloor$ if and only if $p \notin \mathbb{Z}$. If $p \in \mathbb{Z}$ then $\lfloor p \rfloor = p$ and $\lfloor p \rfloor = p-1$.

Proposition 10.3.3.

i) *Let $p > 0$ and let S_p be either ℓ^p or ℓ^{p-} . Then*

$$HC_n^{(q)}(\Gamma^\infty : I_{S_p}) = \begin{cases} 0 & n < q + \lfloor p \rfloor \\ (S_{(p/(\lfloor p \rfloor + 1))}\Omega_{\ell^\infty}^{q-\lfloor p \rfloor})_{\mathcal{E}} & n = q + \lfloor p \rfloor \end{cases}$$

In particular, the first nonzero group is

$$HC_{2\lfloor p \rfloor}(\Gamma^\infty : I_{S_p}) = HC_{2\lfloor p \rfloor}^{[\lfloor p \rfloor]}(\Gamma^\infty : I_{S_p}) = HC_0(\Gamma^\infty : I_{S_{p/(\lfloor p \rfloor + 1)}})$$

which was computed in 9.5.4.

ii)

$$HC_n^{(q)}(\Gamma^\infty : I_{\ell^{p+}}) = \begin{cases} 0 & n < q + [p] \\ HC_{q-[p]}^{q-[p]}(\Gamma^\infty : I_{\ell^{p/([p]+1)}}) & n = q + [p] \end{cases}$$

In particular, the first nonzero group is

$$HC_{2[p]}(\Gamma^\infty : I_{\ell^{p+}}) = HC_{2[p]}^{([p])}(\Gamma^\infty : I_{\ell^{p+}}) = HC_0(\Gamma^\infty : I_{\ell^{(p/([p]+1))+}}) = \mathbb{C}$$

Proof. This is a straightforward application of the spectral sequence of Corollary 10.2.6 and Lemma 10.3.2 and Proposition 10.1.6. \square

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